Admissible Bases via Stable Canonical Rules

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Multi Conclusion Rules

We review a recent application of multi-conclusion rules, giving a new proof for decidability of rule admissibility in **IPC**, **K4**, **S4**.

Rule admissibility in these systems is a problem having a long history.

For **IPC**:

- Friedman 1975 (raises the problem);
- Rybakov 1984 (first solution);
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Why many conclusions?

A *multiple-conclusion rule* is a pair of finite sets of formulae $\langle \Gamma, S \rangle$.

If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}, S = \{\delta_1, \ldots, \delta_m\}$, we write the rule $\langle \Gamma, S \rangle$ as Γ / S or as

$$\frac{\gamma_1, \ \ldots, \ \gamma_n}{\delta_1 \mid \cdots \mid \delta_m} \ (R)$$

The formulae $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ are said to be the *premises* of the rule (R) and the formulae $S = \{\delta_1, \ldots, \delta_m\}$ are said to be the *conclusions* of the rule (R).

The rule (R) is *valid* in a modal algebra $(A, \Diamond)$ iff for every valuation V

$$V(\gamma_1) = 1 \ \& \ \cdots \ \& \ V(\gamma_n) = 1 \quad \Rightarrow \quad V(\delta_1) = 1 \text{ or } \cdots \text{ or } V(\delta_m) = 1 \ .$$

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Multiple-conclusion rules recently gained attention in the literature from many points of view.

From an algebraic and semantic point of view (Kracht 07, Jerabek 09, N. & G. Bezhanishvili & Iemhoff 2014), they constitute an essential tool for investigating classes of algebras beyond varieties and they supply nice axiomatizations.

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Derived Rules

Let $\mathcal{R}$ be a set of multiple-conclusion rules; a multiple-conclusion rule Γ/S is *derivable from* $\mathcal{R}$, written $\mathcal{R} \vdash \Gamma/S$, iff every modal algebra validating all rules in $\mathcal{R}$ also validates Γ/S.

In the terminology of modal rule systems[1] (Jerabek 09, N. & G. Bezhanishvili & Iemhoff 2014), it can be proved that this equivalently means that Γ/S belongs to the smallest modal rule system including $\mathcal{R}$.

A *Hilbert style* calculus for recognizing $\mathcal{R} \vdash \Gamma/S$ is built in (N. Bezhanishvili & Ghilardi 2014).

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Stability

A *stable embedding* of a modal algebra $\mathfrak{A} = (A, \Diamond)$ into a modal algebra $\mathfrak{B} = (B, \Diamond)$ is an injective Boolean morphism $\mu : A \to B$ such that we have $\Diamond\mu(x) \leq \mu(\Diamond x)$ for all $x \in A$.

A class $\mathcal{C}$ of modal algebras is said to be *stable* iff whenever $\mathfrak{B} \in \mathcal{C}$ and $\mathfrak{A}$ has a stable embedding into $\mathfrak{B}$, then $\mathfrak{A} \in \mathcal{C}$ too.

We have dual notions for general frames. $\mathfrak{F} = (W, R, P)$ is a *stable image* of $\mathfrak{F}' = (W', R', P')$ iff there is a continuous (i.e. $S \in P \Rightarrow f^{-1}(S) \in P'$) surjective map $f : W' \to W$ such that xRy implies $f(x)R'f(y)$ for all $x, y \in W'$.

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Stable Canonical Rules

Given a domain (i.e. clopen) $d \subseteq W'$, we say that a stable map f from $\mathfrak{F} = (W, R, P)$ into $\mathfrak{F}' = (W', R', P')$ satisfies the closed domain condition for d iff $f^{-1}(\Diamond d) = \Diamond f^{-1}(d)$ i.e. iff for all x

$$d \cap {\uparrow} f(x) \neq \varnothing \Rightarrow d \cap f({\uparrow} x) \neq \varnothing.$$

We introduce now a class of rules, called ‘stable canonical rules’, see N. & G. Bezhanishvili, R. Iemhoff (2014). No transitivity is assumed.

Stable Canonical Rules

Definition

Let $\mathfrak{F} = (F, R)$ be a finite frame and $\mathfrak{D}$ be a set of domains in F; the stable canonical rule $\rho(\mathfrak{F}, \mathfrak{D})$ is the multi-conclusion rule:

$$\frac{\bigvee_{i=1}^{n} x_{a_i},\ \{\delta_{ij} \mid i \neq j\},\ \{x_{a_i} \to \Box \bigvee_{a_i R b} x_b\}_i,\ \{\phi_d \mid d \in \mathfrak{D}\}}{\neg x_{a_1} \mid \cdots \mid \neg x_{a_n}}$$

where we suppose that $F = \{a_1, \ldots, a_n\}$ and

- $\delta_{ij} := \neg(x_{a_i} \wedge x_{a_j})$;
- $\phi_d := \bigwedge_i \bigwedge_{b \in d, a_i R b}(x_b \to \Diamond x_b)$.

Completeness

Proposition

A general frame (W, R, P) refutes $\rho(\mathfrak{F}, \mathfrak{D})$ iff there is a stable surjective map from (W, R, P) onto $\mathfrak{F} = (F, R_F)$ satisfying the closed domain condition for all $d \in \mathfrak{D}$.

We have a completeness result here (without transitivity hypothesis):

Theorem (N. & G. Bezhanishvili & Iemhoff 2014)

Given a rule Γ/S one can always find a finite set of stable canonical rules equivalent to it over **K**.

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Fmp

A stable rule is a stable canonical rule of the kind $\rho(\mathfrak{F}, \emptyset)$. A modal calculus K is *stable* iff so is the class of modal algebras validating it (equivalently: the class of descriptive frames validating it).

Theorem (N. & G. Bezhanishvili & Iemhoff 2014)

(i) *A modal calculus K is stable iff it is axiomatizable via stable rules.*

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Fmp and Bpp

To get better proof-theoretic properties, rule $\rho(\mathfrak{F}, \emptyset)$ is modified into the rule $\rho^+(\mathfrak{F}, \emptyset)$ below:

$$\frac{\bigvee_{i=1}^{n} x_{a_i}, \quad \bigwedge_{i \neq j} \neg(x_{a_i} \wedge x_{a_j}), \quad \bigwedge_{i=1}^{n}(x_{a_i} \to \Box r_{a_i}), \quad \bigwedge_{i=1}^{n}(r_{a_i} \to \bigvee_{b \in R_F(a_i)} x_b)}{\neg x_{a_1} \mid \cdots \mid \neg x_{a_n}}$$

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Rules $\rho(\mathfrak{F}, \emptyset)$ and $\rho^+(\mathfrak{F}, \emptyset)$ are inter-derivable.

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Theorem (N. B. & S. G. 2014)

Any modal calculus axiomatized by rules of the kind $\rho^+(\mathfrak{F}, \emptyset)$ enjoys bounded proof property (bpp) and fmp.

Corollary

Let $\mathcal{C}$ be a stable class of (ordinary) Kripke frames such that membership of a finite frame in $\mathcal{C}$ is decidable. Then validity of a formula (more generally, of a rule) in $\mathcal{C}$ is decidable as well.

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Dichotomy property

The following result was established in the context of the multi-conclusion reformulation of canonical rules in the sense of M. Zakharyaschev:

Theorem (Jerabek 2009)

Over various common logics (including **K4**, **S4**, **GL**, ...*), a canonical rule is either admissible or equivalent to an assumption-free rule.*

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Admissible Rules

We let $(S^m_{n,\ell})$ be the rule

$$\frac{\bigwedge_{l=1}^{\ell}(\Box x_l \to x_l) \wedge \bigwedge_{k=1}^{m} \Box(r_k \to \Box(r_k \vee \Box^+ q)) \to \bigvee_{i=1}^{n} \Box p_i}{\Box^+ q \to p_1 | \dots | \Box^+ q \to p_n} \quad (1)$$

and (T_m) be the rule

$$\frac{\bigwedge_{k=1}^{m}(\Diamond r_k \to \Diamond(r_k \wedge \Box^+ q)) \to \bigvee_{i=1}^{n} \Box p_i}{\Box^+ q \to p_1 | \dots | \Box^+ q \to p_n} \quad (2)$$

Proposition

The rules $(S^m_{n,\ell})$ are admissible in **K4** *for all $n, m, \ell \in \omega$, and the rules (T_m) are admissible in* **K4** *for all $m \in \omega$.*

A Semantic Ingredient

From now on, all frames are assumed to be transitive.

Definition

A stable canonical rule $\rho(\mathfrak{F}, \mathfrak{D})$ is called *r-trivial* if for every $S \subseteq W$, there is a reflexive $w^\circ \in W$ such that

- $S \subseteq R[w^\circ]$; and
- for all $U \in \mathfrak{D}$, if $U \cap R[w^\circ] \neq \varnothing$, then $U \cap (\{w^\circ\} \cup R^+[S]) \neq \varnothing$.

A stable canonical rule $\rho(\mathfrak{F}, \mathfrak{D})$ is called *u-trivial* if for every $S \subseteq W$, there is $w^\bullet \in W$ such that

- $S \subseteq R[w^\bullet]$; and
- for all $U \in \mathfrak{D}$, if $U \cap R[w^\bullet] \neq \varnothing$, then $U \cap R^+[S] \neq \varnothing$.

A stable canonical rule is *trivial* if it is both r-trivial and u-trivial.

Dichotomy

The intuitionistic version of r-triviality is also used in J. Goudsmit's thesis under the name of 'adequate extendability'.

The dichotomy property can now be stated as follows.

Theorem

The following are equivalent:

1. $\rho(\mathfrak{F}, \mathfrak{D})$ *is admissible.*
2. $\rho(\mathfrak{F}, \mathfrak{D})$ *is derivable from* $\{S^m_{n,\ell} : m, n, \ell \in \omega\} \cup \{T_m : m \in \omega\}$.
3. $\rho(\mathfrak{F}, \mathfrak{D})$ *is not trivial.*
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Proof of (3) $\Rightarrow$ (2) (Sketch): if $\rho(\mathfrak{F}, \mathfrak{D})$ is not derivable from $\{S^m_{n,\ell} : m, n, \ell \in \omega\} \cup \{T_m : m \in \omega\}$, by algebraic completeness, there is a transitive descriptive frame (W, R, P), where these rules are valid and $\rho(\mathfrak{F}, \mathfrak{D})$ fails.

Then there is a continuous stable morphism from W onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$.

This morphism and the shapes of $S^m_{n,\ell}$, T_m are used to build the points $x^\bullet$, x° required by the triviality condition.

Dichotomy

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Proof of (4) $\Rightarrow$ (3) (Sketch): this is the most difficult point.

We need a Lemma saying that a rule $\rho(\mathfrak{F}, \mathfrak{D})$ is equivalent to an assumption-free rule iff every stable morphism from a clopen upset of a transitive descriptive frame (W, R, P) onto $\mathfrak{F}$ that satisfies (CDC) for $\mathfrak{D}$ can be extended to the whole of W.

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Admissible Bases

Corollary

The rules $\{S^m_{n,\ell} : m, n \in \omega\} \cup \{T_m : m \in \omega\}$ *form an admissible basis for* **K4**.

The admissible basis $\{S^m_{n,\ell} : m, n \in \omega\} \cup \{T_m : m \in \omega\}$ is equivalent to known admissible bases. Since rule admissibility is Π^0_1 and derivability from a recursive set of rules is Σ^0_1, we get:

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Admissibility of inference rules in **K4** *is decidable.*

A more practical procedure would compute, for a given rule, a set of stable canonical rules equivalent to it and check triviality for each of them.

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Little modifications adjust the above results to **S4** and to **IPC**. For **S4**, we just take out the rules T_m. We give few more details for **IPC**.

Let (G_n) be the rule (this is a version of Visser's n-th rule):

$$\frac{(\bigvee_{i=1}^{n} p_i \to q) \to \bigvee_{i=1}^{n} p_i}{q \to p_1 | \ldots | q \to p_n} \tag{3}$$

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The following are equivalent:

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3. $\gamma(\mathfrak{F}, \mathfrak{D})$ *is not r-trivial.*
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Rule admissibility in **IPC** *is decidable; the rules* $\{G_n : n \in \omega\}$ *form an admissible basis.*

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Conclusions

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