Topological clones

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Outline

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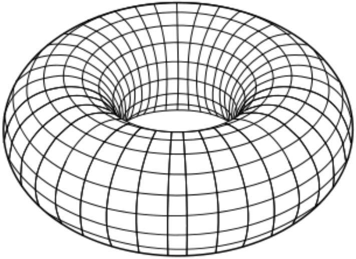

I: Abstract clones

Algebras, function clones

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Clo($\mathfrak{A}$) is a function clone:

- closed under composition: $f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$;
- contains projections $\pi_i^n(x_1, \ldots, x_n) = x_i$.

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Here: algebras up to "clone equivalence".

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- preserves arities;
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We write $\mathcal{C} \to \mathcal{D}$ if there exists a clone homomorphism from $\mathcal{C}$ to $\mathcal{D}$.

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- $\mathcal{D} \in \mathsf{HSP}(\mathcal{C})$;
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What about $\mathsf{HSP}^{\mathsf{fin}}$ of infinite function clones?

Analogy with groups and monoids

Permutation group	Abstract group
Transformation monoid	Abstract monoid
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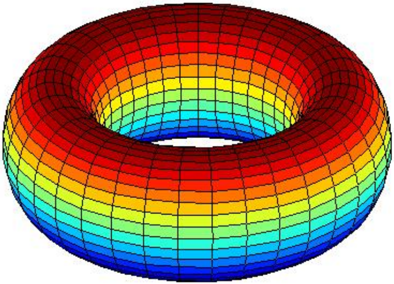

II: Topological clones

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For finite function clones: topology discrete.

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- $\mathcal{C} \to \mathcal{D}$ *surjectively + uniformly continuously.*

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Two closed function clones which are isomorphic, but not topologically.
(Bodirsky + Evans + Kompatscher + MP 2015)

C			4		3		2	8			9				B
7						A				6			4		
	E		8	D				F		5	2		C	7	
			0		7				B		D		6		E
4				9							E		1		
	6		2							0		5			3
	0	B	1	4		2			9				E		
	9	5			A	B	C	6			7				
	C		B		6		F	A	2		5			0	4
A		2			5	D	0			C	8	3	B		1
		0	F	B								D		2	
5			3		8				1		0	9	F		
3	8			5		6	E	0		F				9	
		C		F		1						B		E	
0							8				6	7			D
		4		A	D		7		E		C	2			5

III: pp interpretations, Constraint Satisfaction Problems

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Observe: Pol(Γ) $\supseteq$ End(Γ) $\supseteq$ Aut(Γ).

Closed polymorphism clones

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Let Γ, Δ be relational structures.

$$\text{What does } \mathrm{Pol}(\Delta) \in \mathrm{HSP}^{\mathrm{fin}}(\mathrm{Pol}(\Gamma)) \text{ imply for } \Gamma, \Delta?$$

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- Δ *has a pp interpretation in* Γ*:*
  it is a pp-definable homomorphic image
  of a pp-definable subuniverse
  of a finite power
  of a structure which is pp-definable in Γ.

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Let Γ be countable ω-categorical or finite, and Δ be finite. TFAE:

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Corollary (Bodirsky + MP '11)

Let Γ be countable ω-categorical or finite. TFAE:

- $\mathrm{Pol}(\Gamma) \to \mathbf{1}$ *continuously;*
- *All finite structures have a pp interpretation in* Γ.

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Γ is called the template of the CSP.

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Let Γ be a structure in a finite relational language.

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Corollary

Let Γ be finite or countable ω-categorical.

If Pol(Γ) $\to$ $\mathbf{1}$ continuously, then CSP(Γ) is NP-hard.

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Observation (Bulatov + Krokhin + Jeavons 2000)

For every finite structure Γ there is a finite structure $\mathfrak{C}(\Gamma)$ such that

- $f(x, \ldots, x) = x$ for all polymorphisms of $\mathfrak{C}(\Gamma)$
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What does this mean for Pol(Γ)?

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For every ω-categorical structure Γ there is an ω-categorical structure $\mathfrak{C}(\Gamma)$ ("model-complete core of Γ") such that

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IV: Topological clones revisited

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Proposition

Let Γ, Δ be structures, where Γ is ω-categorical. TFAE:

- Δ is homomorphically equivalent to a pp definable structure of Γ
- $\mathrm{Pol}(\Delta)$ contains a double shrink of $\mathrm{Pol}(\Gamma)$.

D, HSP, and weak clone homomorphisms

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$\mathbf{D}(\mathfrak{A})\ldots$ all double shrinks of $\mathfrak{A}$.

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D and HSP^{fin}

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Theorem (Barto + MP 2015)

Let Γ be finite or ω-categorical, let Δ be finite. TFAE:

- Δ *can be obtained from* Γ *by homomorphic equivalence, adding of constants to model-complete cores, and pp interpretations.*
- $\mathrm{Pol}(\Gamma) \rightsquigarrow \mathrm{Pol}(\Delta)$ *uniformly contiuously.*

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Observation: Old $\Longrightarrow$ New.

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- Is there a closed clone $\mathcal{C}$ such that $\mathbf{1} \in \mathrm{HSP}(\mathcal{C})$, but $\mathbf{1} \notin \mathrm{HSP}^{\mathrm{fin}}(\mathcal{C})$?

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- Is there a better name than “double shrink”?

Reference

L. Barto, J. Opršal, and M. Pinsker

The wonderland of the double shrink

In preparation.

Wayne Ferrebee, *Torus with Spearman, Bagpipes and Barnacle*