Canonicity and Relativized Canonicity via Pseudo Correspondence (Unified Correspondence II)

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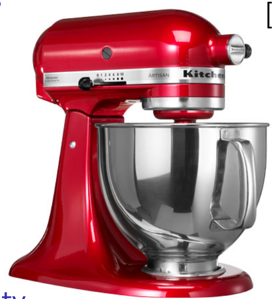
Unified correspondence

Hybrid logics
[CR15]

DLE-logics
[CP12, CPS]

Substructural logics
[CP15]

Mu-calculi
[CFPS15, CGP14, CC15]



Display calculi
[GMPTZ]

Regular DLE-logics
Kripke frames with
impossible worlds
[PSZ15a]

Jónsson-style vs
Sambin-style canonicity
[PSZ15b]

Finite lattices and
monotone ML
[FPS15]

Canonicity via
pseudo-correspondence
[CPSZ]

Canonicity via Correspondence [Sambin-Vaccaro89]

$$
\begin{array}{ccc}
\mathbb{G} \Vdash \varphi & & \mathbb{G}^\sharp \Vdash \varphi \\
\Updownarrow & & \Updownarrow \\
\mathbb{G} \models \mathsf{FO}(\varphi) & \Leftrightarrow & \mathbb{G}^\sharp \models \mathsf{FO}(\varphi)
\end{array}
$$

Canonicity via Pseudo Correspondence [Venema98]

$$\begin{array}{ccc} \mathbb{G} \Vdash \varphi & & \mathbb{G}^\sharp \Vdash \varphi \\ \Downarrow & & \Uparrow \\ \mathbb{G} \models \mathrm{FO}(\varphi) & \Leftrightarrow & \mathbb{G}^\sharp \models \mathrm{FO}(\varphi) \end{array}$$

where $\varphi := \pi(p \vee q) \leftrightarrow \pi(p) \vee \pi(q)$
and $\pi(x)$ positive in x

Definition: φ modal formula, α f.o. sentence *pseudo-correspond* if for any DGF $\mathbb{G}$ and any Kripke frame $\mathbb{F}$,

1. if $\mathbb{G} \Vdash \varphi$, then $\mathbb{G}^\sharp \models \alpha$; ($\Downarrow + \Leftrightarrow$)
2. if $\mathbb{F} \models \alpha$ then $\mathbb{F} \Vdash \varphi$. ($\Uparrow$)

Correspondence & Canonicity

Algebraic Pseudo Correspondence via ALBA

$$\mathbb{A} \Vdash \varphi \leq \psi \qquad\qquad \mathbb{A}^\delta \Vdash \varphi \leq \psi$$

$$\Downarrow \qquad\qquad\qquad \Uparrow$$

$$\mathbb{A}^\delta \models_{\mathbb{A}} \mathrm{FO}(\varphi \leq \psi) \iff \mathbb{A}^\delta \models \mathrm{FO}(\varphi \leq \psi)$$

Fact: The following pseudo-correspond for every $\pi(x)$ positive in x:

1. $\pi(p \vee q) \leq \pi(p) \vee \pi(q)$;
2. $\forall \mathbf{m}[\pi(\bot) \leq \mathbf{m} \Rightarrow \pi(\blacksquare_\pi \mathbf{m}) \leq \mathbf{m}]$.

$\blacksquare_\pi$ interpreted as the operation $u \mapsto \blacksquare_\pi u := \bigvee\{i \in J^\infty(\mathbb{A}^\delta) \mid \pi(i) \leq u\}$.

$$\Diamond_\pi \dashv \blacksquare_\pi,$$

where $u \mapsto \Diamond_\pi u := \bigvee\{\pi(j) \mid j \in J^\infty(\mathbb{A}^\delta) \text{ and } j \leq u\}$.

Lemma: For every positive term π and every algebra $\mathbb{A}$,

if $\mathbb{A} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q)$, then $\mathbb{A}^\delta \models_{\mathbb{A}} \pi(p) \leq \Diamond_\pi(p) \vee \pi(\bot)$.

$$
\begin{array}{ll}
 & \mathbb{A}^\delta \models_{\mathbb{A}} \forall p[\pi(p) \leq \Diamond_\pi(p) \vee \pi(\bot)] \\
\text{iff} & \mathbb{A}^\delta \models_{\mathbb{A}} \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(p) \ \& \ \Diamond_\pi(p) \vee \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \mathbb{A}^\delta \models_{\mathbb{A}} \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(p) \ \& \ \Diamond_\pi(p) \leq \mathbf{m} \ \& \ \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \mathbb{A}^\delta \models_{\mathbb{A}} \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(p) \ \& \ p \leq \blacksquare_\pi \mathbf{m} \ \& \ \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \mathbb{A}^\delta \models_{\mathbb{A}} \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(\blacksquare_\pi \mathbf{m}) \ \& \ \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \mathbb{A}^\delta \models \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(\blacksquare_\pi \mathbf{m}) \ \& \ \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \mathbb{A}^\delta \models \forall \mathbf{m}[\pi(\bot) \leq \mathbf{m} \Rightarrow \pi(\blacksquare_\pi \mathbf{m}) \leq \mathbf{m}].
\end{array}
$$

Lemma: For every positive term π and every algebra $\mathbb{A}$,

$$\text{if } \mathbb{A} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q), \quad \text{then} \quad \mathbb{A}^\delta \models_{\mathbb{A}} \pi(p) \leq \Diamond_\pi(p) \vee \pi(\bot).$$

$$\begin{array}{ll}
 & \mathbb{A}^\delta \models_{\mathbb{A}} \forall p[\pi(p) \leq \Diamond_\pi(p) \vee \pi(\bot)] \\
\text{iff} & \mathbb{A}^\delta \models_{\mathbb{A}} \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(p) \;\&\; \Diamond_\pi(p) \vee \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \mathbb{A}^\delta \models_{\mathbb{A}} \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(p) \;\&\; \Diamond_\pi(p) \leq \mathbf{m} \;\&\; \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \mathbb{A}^\delta \models_{\mathbb{A}} \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(p) \;\&\; p \leq \blacksquare_\pi \mathbf{m} \;\&\; \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \mathbb{A}^\delta \models_{\mathbb{A}} \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(\blacksquare_\pi \mathbf{m}) \;\&\; \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \mathbb{A}^\delta \models \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(\blacksquare_\pi \mathbf{m}) \;\&\; \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \mathbb{A}^\delta \models \forall \mathbf{m}[\pi(\bot) \leq \mathbf{m} \Rightarrow \pi(\blacksquare_\pi \mathbf{m}) \leq \mathbf{m}].
\end{array}$$

Prop: For every positive term π and every perfect algebra $\mathbb{B}$,

$$\text{if } \mathbb{B} \models \forall \mathbf{m}[\pi(\bot) \leq \mathbf{m} \Rightarrow \pi(\blacksquare_\pi \mathbf{m}) \leq \mathbf{m}], \quad \text{then} \quad \mathbb{B} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q).$$

The assumption is ALBA-equivalent to $\mathbb{B} \models \pi(p) \leq \Diamond_\pi(p) \vee \pi(\bot)$. Therefore ($\pi$ positive) $\mathbb{B} \models \pi(p) = \Diamond_\pi(p) \vee \pi(\bot)$. Hence π is completely additive, so $\mathbb{B} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q)$, as required.

Φ-Meta inductive inequalities

consider the language with placeholder connectives $\{\dot{\Diamond}, \boxdot, \dot{\lhd}, \dot{\rhd}\}$.

$$\Phi : \{\dot{\Diamond}, \boxdot, \dot{\lhd}, \dot{\rhd}\} \to \{\pi, \sigma, \lambda, \rho\}$$

$$\dot{\Diamond} \mapsto \pi \qquad \boxdot \mapsto \sigma$$

$$\dot{\lhd} \mapsto \lambda \qquad \dot{\rhd} \mapsto \rho$$

An inequality $\varphi \leq \psi$ is **Φ-meta inductive** if

$$\varphi \leq \psi = (s \leq t)[\Phi(\odot)/\odot]$$

for some inductive $\{\dot{\Diamond}, \boxdot, \dot{\lhd}, \dot{\rhd}\}$-inequality $s \leq t$.

Example of meta inductive: $\Diamond\Box\Diamond\Box p \leq \Box\Diamond\Box\Diamond p$
$s \leq t := \dot{\Diamond}\boxdot p \leq \boxdot\dot{\Diamond} p$, which is Sahlqvist
$\dot{\Diamond} \mapsto \Diamond\Box\Diamond$, $\boxdot \mapsto \Box$.

Relativized canonicity of Φ-meta inductive inequalities

Proposition:

$$\Phi : \{\Diamond, \boxdot, \lhd, \rhd\} \to \{\pi, \sigma, \lambda, \rho\}$$

$$\Diamond \mapsto \pi \qquad \boxdot \mapsto \sigma$$

$$\lhd \mapsto \lambda \qquad \rhd \mapsto \rho$$

If

$$\mathbb{A} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q) \quad \mathbb{A} \models \sigma(p) \wedge \sigma(q) \leq \sigma(p \wedge q)$$

$$\mathbb{A} \models \rho(p \vee q) \leq \rho(p) \wedge \rho(q) \quad \mathbb{A} \models \lambda(p) \vee \lambda(q) \leq \lambda(p \wedge q),$$

then

$$\mathbb{A} \models \varphi \leq \psi \ \Rightarrow \ \mathbb{A}^{\delta} \models \varphi \leq \psi$$

for any Φ-meta inductive inequality $\varphi \leq \psi$.

Proof strategy

$$
\begin{array}{ccc}
\mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\
\Updownarrow & & \\
\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi & & \Updownarrow \\
\Updownarrow & & \\
\mathbb{A}^\delta \models_{\mathbb{A}} \mathsf{ALBA}^e(\varphi \leq \psi) & \Leftrightarrow & \mathbb{A}^\delta \models \mathsf{ALBA}^e(\varphi \leq \psi)
\end{array}
$$

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