Duality for Non-monotonic Consequence Relations and Antimatroids

Johannes Marti and Riccardo Pinosio

ILLC, University of Amsterdam

June 22, 2015

Overview

Non-monotonic consequence relations over Boolean algebras are dual to antimatroids over Stone spaces.

Non-monotonic consequence relations

A Non-monotonic consequence relation is a binary relation $\mid\sim$ over the elements of a Boolean algebra such that:

- $a \mid\sim a$ (Id)
- If $a \mid\sim c$ and $c \leq d$ then $a \mid\sim d$ (RW)
- If $a \mid\sim c$ and $a \mid\sim d$ then $a \mid\sim c \wedge d$ (And)
- If $a \mid\sim b \wedge c$ then $a \wedge b \mid\sim c$ (WCM)
- If $a \mid\sim c$ and $b \mid\sim c$ then $a \vee b \mid\sim c$ (Or)

This is the non-nested fragment of many conditional logics.

Order Semantics

Let $\leq$ be a poset over some set

$$A \mathrel{|\!\sim} C \quad \text{iff} \quad \mathrm{Min}_{\leq}(A) \subseteq C \quad \text{iff} \quad A \subseteq \mathrm{cl}(A \cap C)$$

Antimatroids

An antimatroid is a closure operator cl on some set which satisfies:

If $x \neq y$ and $x, y \notin \mathrm{cl}(U)$ then $x \notin \mathrm{cl}(U \cup \{y\})$ or $y \notin \mathrm{cl}(U \cup \{x\})$

The upwards-closure in a poset is an antimatroid.

Antimatroids can be represented by posets.

The cl-closed sets are called convex.

Complements of convex sets are called feasible.

Finite duality

Non-monotonic consequence relations over a finite Boolean algebra correspond to antimatroids over finite sets.

$$A \mathrel{|\!\sim} C \quad \text{iff} \quad A \subseteq \mathsf{cl}(A \cap C)$$

$$\mathsf{cl}(D) = \bigcup \{B \mid B \mathrel{|\!\sim} D\}$$

Extending to the Stone-topology

$$A \mathrel{|\!\!\sim} C \quad \text{iff} \quad A \subseteq \mathrm{cl}(A \cap C)$$

$$\mathrm{cl}(X) = \bigcap \left\{ \bigcup \{B \mid B, D \text{ clopen } B \mathrel{|\!\!\sim} D, D \subseteq U\} \; U \text{ open}, \; X \subseteq U \right\}$$

For a duality we need the following topological conditions on cl:

1. The convex closure of an clopen is open.
2. The convex closure of an open is the union of the convex closures of the clopens contained in it.
3. Every convex set is the intersection of convex open sets.

Further work

- find nicer topological conditions
- investigate topological antimatroids
- interpret the conditional on general topological spaces