TACL 2015, Ischia

$\star$-autonomous categories and tensor products

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An history of Linear Logic

Syntax

λ-calculus

Semantic

Normal functors

An history of Linear Logic

Syntax

λ-calculus

Linearity

A vectorial setting

$f : A \multimap B$ linear

$g : !A \multimap B$ non-linear

Semantic

Normal functors

An history of Linear Logic

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Linear Logic
Girard 88

Linearity
A vectorial setting

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Linear Logic, two implications

Grammar : $A, B ::= 1 | \bot | \top | 0 | A \parr B | A \otimes B | A \oplus B | A \& B | !A | ?A$

U

Cartesian closed category :
Non-linear functions
$!A \multimap B$, $\&$, $\oplus$

Monoidal Closed Category :
Linear Functions
$A \multimap B$, $\otimes$, $\parr$

!

Linear Logic, a linear negation

A model of Linear Logic must also be a *-autonomous category.

It is a monoidal closed category with a distinguished object $\perp$, where the morphism

$$d_A : A \rightarrow (A \multimap \perp) \multimap \perp$$

is an isomorphism.

d_A is the transpose of

$$eval_A : A \otimes (A \multimap \perp) \rightarrow \perp.$$

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Relational model

Formulas as sets, Proofs as relations

Köthe spaces

Ehrhard 02

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Relational model

Differential Linear Logic

Ehrhard Regnier 03

Differentiation

A smooth setting

Köthe spaces

Differential λ-calculus

What do we want

I want to explain to my applied math colleague
what is a model of LL.

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The product

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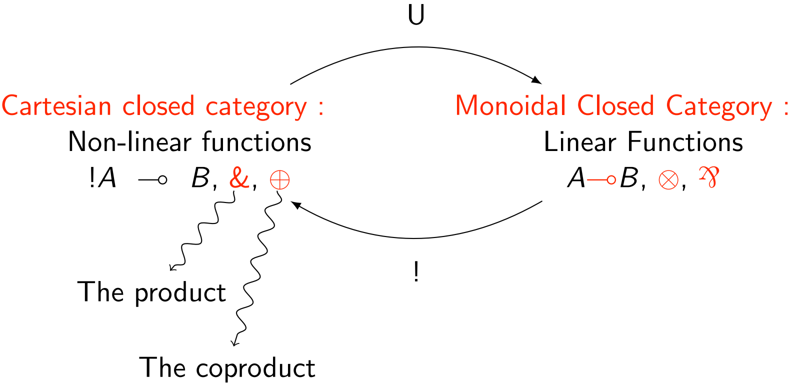

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The product

The coproduct

The tensor product

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I want to explain to my applied math colleague what is a *-autonomous category:

The following must be an isomorphism for every A :

$$d_A : A \to (A \multimap \bot) \multimap \bot$$

d_A is the transpose of

$$\begin{aligned} eval_A : & A \otimes (A \multimap \bot) \to \bot \\ & A \times \mathcal{L}(A, \mathbb{K}) \to \mathbb{K} \\ & x, f \mapsto f(x) \end{aligned}$$

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$$\begin{aligned} d_A : & A \to (A \multimap \bot) \multimap \bot \\ & A \to \mathcal{L}(\mathcal{L}(A, \mathbb{K}), \mathbb{K}) \\ & x \mapsto (\delta_x : f \mapsto f(x)) \end{aligned}$$

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Exclamation

Well, this is a just a category of reflexive vector space.

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Exclamation

Well, this is a just a category of reflexive vector space.

Disapointment

Well, the category of reflexive topological vector space is not closed.

Weak topologies

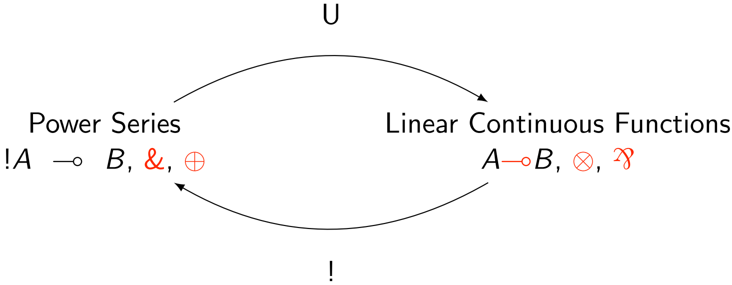
Theorem

The category of spaces endowed with their weak topology is a model of Linear Logic

If the dual E' of a topological vector space E is endowed with its weak topology, then E'' is isomorphic to E.*

The reversible connectives are exactly those preserving the weak topology .

A topology on the algebraic constructions



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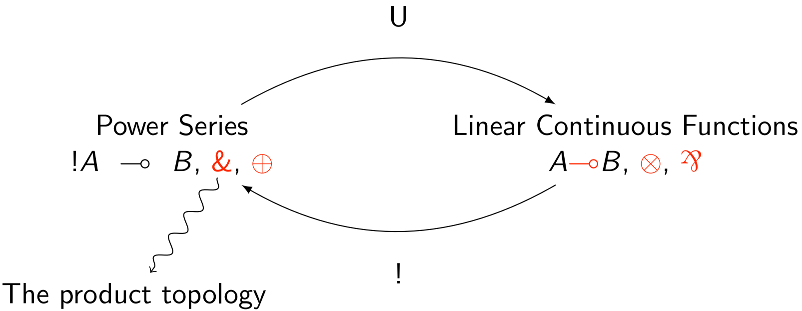

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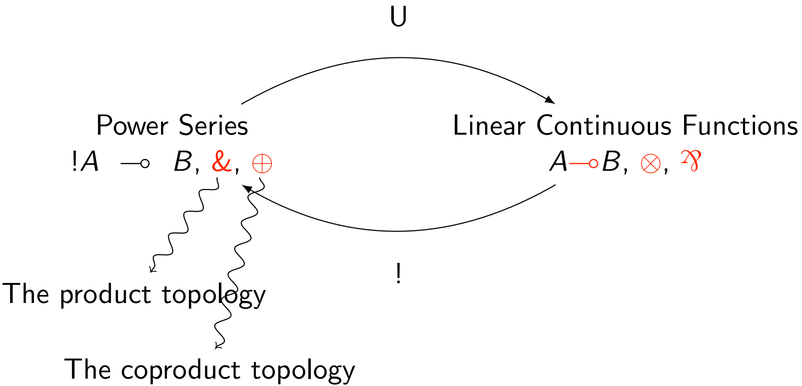

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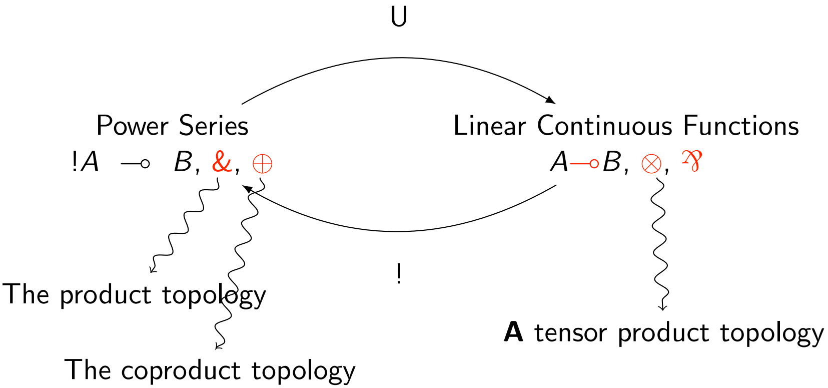

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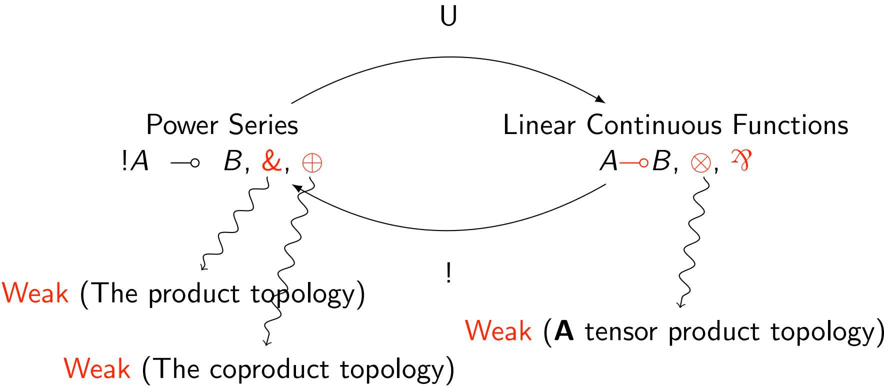

A choice for the tensor product

There are three canonical topologies on the tensor product of two topological vector spaces E and F.

$$E \otimes_i F, E \otimes_\pi F, E \otimes_\epsilon F$$

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- Identifying $\otimes_\pi$ and $\otimes_\epsilon$ defines Nuclear spaces.

A choice for the tensor product

There are three canonical topologies on the tensor product of two topological vector spaces E and F.

$$E \otimes_i F, E \otimes_\pi F, E \otimes_\epsilon F$$

- Identifying $\otimes_\pi$ and $\otimes_\epsilon$ defines Nuclear spaces.
- Fréchet spaces are the complete metrizable spaces. In such a space, $\otimes_\pi$ and $\otimes_i$ correspond.

Nuclear Fréchet spaces are Reflexive spaces

Theorem

A Nuclear space which is also Fréchet or (DF) is reflexive.

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The category of Nuclear Fréchet or (DF) is monoidal closed.

Nuclear Fréchet or (DF) spaces preserve the cartesian product and coproduct.

Nuclear Fréchet spaces are Reflexive spaces

Theorem

A Nuclear space which is also Fréchet or (DF) is reflexive.

> *The category of Nuclear Fréchet or (DF) is monoidal closed.*
>
> *Nuclear Fréchet or (DF) spaces preserve the cartesian product and coproduct.*

Theorem

Nuclear Fréchet (or (DF)) spaces form a model of Polarized Multiplicative Additive Linear Logic.

A smooth Exponential ?

Examples of Nuclear Fréchet or (DF) space :

$$\mathcal{C}_c^\infty(U),\ \mathcal{D}'(U),\ \mathcal{C}^\infty(V),\ \mathcal{H}(V).$$

where U is an open subset of $\mathbb{R}^n$ and V is a smooth or analytical manifold.

They verify :

$$\mathcal{F}'(V)\hat{\otimes}\mathcal{F}'(U) = \mathcal{F}'(U \times V)$$

Thank you.