Quantum logics as relational monoids

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- Every effect algebra is a monoid in **Rel**.
- Some well-known definitions in the theory of effect algebras, appear to come from 2-categorial structure of **RelMon**.
- Maybe I will sketch some related result concerning test spaces.
- Almost everything here has almost nonexistent proof.

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- Objects: sets.
- Morphisms: binary relations; $f : A \to B$ in **Rel** is a set of pairs $f \subseteq A \times B$.
- Identities: $id_A : A \to A$ is the identity relation.
- Composition: if $f : A \to B$ and $g : B \to C$, then $(a, c) \in g \circ f$ iff there exists $b \in B$ such that $(a, b) \in f$ and $(b, c) \in g$.

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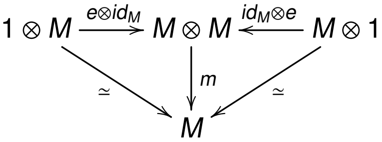
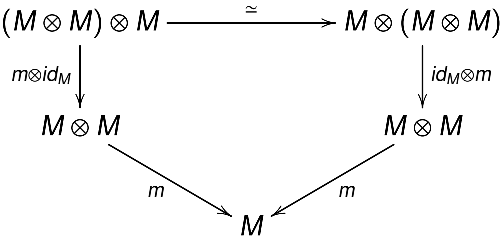
- However, $\times$ is not the product in **Rel**, because...
- ...disjoint union $\sqcup$ is product and, at the same time, coproduct in **Rel**.

Monoids in a monoidal category

Recall, that a monoid is a monoidal category $(\mathcal{C}, \otimes, 1)$ is a triple (M, m, e), where M is an object of $\mathcal{C}$, $m : M \otimes M \to M$ and $e : 1 \to M$ are arrows such that the diagrams





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- Monoids in the monoidal category of complete join semilattices $(\mathbf{Sup}, \otimes, 2)$ are quantales.
- Monoids in the monoidal category of ordinary monoids $(\mathbf{Mon}, \times, 1)$ are commutative monoids.

Monoids in **Rel**

So a monoid in the monoidal category $(\mathbf{Rel}, \times, 1)$ consists of

- a set M,
- a relation $e : 1 \to M$ and
- a relation $* : M \times M \to M$.

such that some diagrams commute.
We call these objects relational monoids

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- e is (essentially) a subset of the underlying set, rather than an element;
- it is misleading to write $a * b = c$ to express the fact that (a, b) is in the relation $*$ with c;
- we write $(a, b) \overset{*}{\mapsto} c$ instead.

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- Small categories are monoids in **Rel**:
  - elements are arrows,
  - the $e : 1 \to M$ is the selection of identity arrows.

Morphisms of monoids in **Rel**

The class of monoids in a monoidal category comes equipped with a standard notion of morphism:

$$\begin{array}{ccc} M \times M & \xrightarrow{h \times h} & M' \times M' \\ {\scriptstyle *}\downarrow & & \downarrow{\scriptstyle *} \\ M & \xrightarrow{\quad h \quad} & M' \end{array}$$

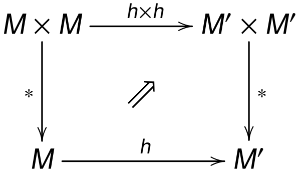
homewer, this notion does not work in the examples we are interested in.

Rel as a 2-category

- A relation $f \subseteq A \times B$ is a set of pairs, so
- every homset $\mathbf{Rel}(A, B)$ is a poset under $\subseteq$.
- That means, that **Rel** is enriched in **Pos**, in other words
- **Rel** a (locally posetal/thin) 2-category.

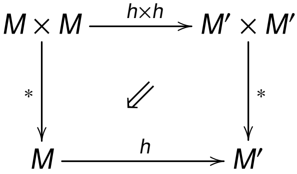
Morphisms of monoids in **Rel**

There are several meaningful notions of morphisms of monoids in **Rel**. In this talk, we shall deal with two of them.



Lax morphism

$$(h \circ *) \subseteq (* \circ (h \times h))$$
$$h \circ e \subseteq e$$



Oplax morphism

$$(h \circ *) \supseteq (* \circ (h \times h))$$
$$e \subseteq h \circ e$$

Category of relational monoids **RelMon**

By Category of relational monoids we mean a 2-category

- 0-cells are relational monoids,
- 1-cells are lax morphisms of relational monoids,
- 2-cells are the $\subseteq$ of relations, inherited from **Rel**.

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- 2-cells are the $\subseteq$ of relations, inherited from **Rel**.

- The category of small categories is a 1-subcategory of this category.
- The category of effect algebras is a subcategory of this category.

Effect algebras

An <u>effect algebra</u> (Foulis and Bennett [1994], Kôpka and Chovanec [1994], Giuntini and Greuling [1989]) is a partial algebra $(E; +, 0, 1)$ with a binary partial operation $+$ and two nullary operations $0, 1$ such that $+$ is commutative, associative and the following pair of conditions is satisfied:

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a + a'$ exists and $a + a' = 1$.

(E4) If $a + 1$ is defined, then $a = 0$.

The $+$ operation is then cancellative and 0 is a neutral element.

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- Because **RelMon** is a 2-category, so it has a lot of structure.
- We can take the standard definitions of 2-categorial things from **RelMon** and examine what they mean for effect algebras.
- We rediscover well-known notions, but now we know where they are coming from.

Easy and nice

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- $\leq : E \to E$ is a left Kan extension of $+$ along the projection $p_1 : E \times E \to E$.
- An effect algebra E satisfies the Riesz decomposition property iff $\geq$ is an endomorphism of E.

Adjoint pairs of morphisms in **RelMon**

Since **RelMon** is a 2-category, we may speak about adjoint pairs of morphisms in **RelMon**. Unwinding the definition, it turns out every left adjoint in **RelMon** is a mapping.

Let

- A, B be relational monoids,
- $f : A \to B$,
- $g : B \to A$.

Then the morphism f is left adjoint to the morphism g, if and only if f is a mapping and $g = f^{-1}$.

Left adjoints in **RelMon**

From this, we obtain a characterization of left adjoints:

Proposition

A morphism $f : A \to B$ *in* **RelMon** *is a left adjoint if and only if* f *is a mapping and*

- *for all* $b_1, b_2 \in B$ *and* $a \in A$ *such that* $(b_1, b_2) \stackrel{*}{\longmapsto} f(a)$*,*
- *there exist* $a_1, a_2 \in A$ *such that* $b_1 = f(a_1)$*,* $b_2 = f(a_2)$ *and* $(a_1, a_2) \stackrel{*}{\longmapsto} a$*.*

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Theorem

Let A, B be effect algebras, let $f : A \to B$ be a morphism of effect algebras. Then f is a left adjoint in **RelMon** *iff*

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Theorem

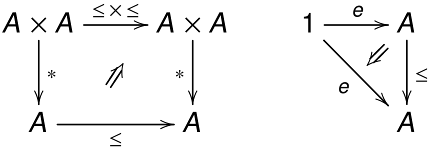
Let A, B be effect algebras, let $f : A \to B$ be a morphism of effect algebras. Then f is a left adjoint in **RelMon** *iff*

- *f is surjective and*
- *the equivalence on A induced by f is an effect algebra congruence in the sense of (Gudder and Pulmannová [1998]).*

Monads in **RelMon**

Since **RelMon** is a 2-category, we may speak about monads in **RelMon**.

A monad in **RelMon** on a relational monoid A can be characterized as a preorder relation $\leq : A \to A$ such that



commute.

Monads arising from adjunctions in 2-categories

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- In every 2-category, an adjoint pair of morphisms gives rise to a monad.
- In **Cat**, every monad arises from an adjoint pair (Eilenberg-Moore, Kleisli).
- But this is not true in every 2-category.
- In particular, in **Rel** the monads (=preorders) arising from adjunctions can be characterized as equivalence relations.

Monads arising from adjunctions in **RelMon**

If a monad $\sim : A \to A$ arises from an adjunction, then

- $\sim$ is an equivalence relation,
- the diagram

$$\begin{array}{ccc} A \times A & \xrightarrow{\sim \times \sim} & A \times A \\ \Big\downarrow {\scriptstyle *} & \nearrow\!\!\!\!\nearrow & {\scriptstyle *} \Big\downarrow \\ A & \xrightarrow[\sim]{\qquad} & A \end{array}$$

  commutes and
- if x is a unit of A and $x \sim y$, then y is a unit of A.

Monads arising from adjunctions in **RelMon**

If the multiplication is actually a partial operation, we obtain another property of a monad arising from an adjunction:

- If $a_1 \sim b_1$, $a_2 \sim b_2$ and both $a_1 * a_2$ and $b_1 * b_2$ exist, then $a_1 * a_2 \sim b_1 * b_2$.

"Dimension equivalences" on effect algebras

For an effect algebra E, we may characterize monads $\sim: E \to E$ arising from adjunctions in **RelMon** as follows:

- $\sim$ is an equivalence.
- If $a_1 \sim b_1$, $a_2 \sim b_2$ and both $a_1 + a_2$ and $b_1 + b_2$ exist, then $a_1 + a_2 \sim b_1 + b_2$.
- If $a \sim b_1 + b_2$, then there are a_1, a_2 such that $a = a_1 + a_2$, $a_1 \sim b_1$, $a_2 \sim b_2$.

$E/\sim$ is then a partial monoid.

Example

- Take a Boolean algebra B; this is an effect algebra with $+$ being the disjoint join.
- Introduce a equivalence on B by the rule

$$a \sim b \Leftrightarrow [0, a] \simeq [0, b]$$

Then this is a dimension equivalence.

A more fancy example

- Let A be an involutive ring with unit, in which

$$x^*x + y^*y = 0 \implies x = y = 0.$$

- Let $P(A)$ be the set of all self-adjoint idempotents in A. For $e, f \in P(A)$, write $e \oplus f = e + f$ iff $ef = 0$, otherwise let $e \oplus f$ be undefined. Then $(P(A); \oplus, 0, 1)$ is an effect algebra.
- For e, f in $P(A)$, write $e \sim f$ iff there is $w \in A$ such that $e = w^*w$ and $f = ww^*$.
- Then this is a dimension equivalence.

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(Dvurečenskij and Pulmannová [2000]) For every cancellative positive partial abelian monoid E and every dimension equivalence $\sim$ *on P,* $P/\sim$ *is a positive partial abelian monoid.*

A new perspective:

- $P/\sim$ is the EM-object for the monad $\sim$.

Comonads

Let E be an effect algebra. A relation $i : E \to E$ is a comonad in **RelMon** iff

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- A subset of a test is an event.

Relations on events

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- Two events a, b are said to be orthogonal (in symbols $a \perp b$) if they are disjoint and their union is an event.
- It is obvious that the set of all events of a test space equipped with the disjoint union of orthogonal events $+$ and $\emptyset$ is a partial commutative monoid.
- If $a \perp b$ and $a \cup b$ is a test, then they are complements of each other (in symbols $a \operatorname{co} b$).

Algebraic test spaces

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- A test space is algebraic if $\mathrm{co} = \mathrm{co} \circ \mathrm{co} \circ \mathrm{co}$.
- For an algebraic test space, the relation $\sim := \mathrm{co} \circ \mathrm{co}$, called perspectivity is an equivalence on events and
- the partial abelian monoid of events, factored by $\sim$ is an orthoalgebra, i.e. and effect algebra with $a \wedge a' = 0$.

A characterization of algebraic test spaces

Proposition

Let $(X, \mathcal{T})$ be a test space. Let us write $(A, +, \emptyset)$ for the partial commutative monoid of the events of $(X, \mathcal{T})$. The following are equivalent.

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2. *$\sim$ is an equivalence relation and an oplax endomorphism of A.*
3. *$\sim$ is a preorder and an oplax endomorphism of A.*
4. *$(A, \tilde{+})$ is associative, where $\tilde{+} = \sim \circ +$.*

Moreover, if $\sim$ is a preorder and $\sim$ is an lax and oplax endomorphism of A, then $A/\sim$ is a Boolean algebra.

Thank you for your attention.

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