Quantum logics as relational monoids

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Every effect algebra is a monoid in $\text{Rel}$. Some well-known definitions in the theory of effect algebras, appear to come from 2-categorial structure of $\text{RelMon}$. Maybe I will sketch some related result concerning test spaces. Almost everything here has almost nonexistent proof.
What?

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The category of sets and relations

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- **Objects**: sets.
- **Morphisms**: binary relations; $f : A \to B$ in $\text{Rel}$ is a set of pairs $f \subseteq A \times B$. 
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...denoted by \textbf{Rel}.

- Objects: sets.
- Morphisms: binary relations; $f : A \rightarrow B$ in \textbf{Rel} is a set of pairs $f \subseteq A \times B$.
- Identities: $id_A : A \rightarrow A$ is the identity relation.
- Composition: if $f : A \rightarrow B$ and $g : B \rightarrow C$, then $(a, c) \in g \circ f$ iff there exists $b \in B$ such that $(a, b) \in f$ and $(b, c) \in g$. 
**Rel** is a monoidal category

- The usual direct product of sets \( \times : \textbf{Rel} \times \textbf{Rel} \to \textbf{Rel} \) is a bifunctor...

- However, \( \times \) is not the product in \( \textbf{Rel} \), because...

- Disjoint union \( \sqcup \) is product and, at the same time, coproduct in \( \textbf{Rel} \).
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- ...disjoint union $\sqcup$ is product and, at the same time, coproduct in Rel.
Monoids in a monoidal category

Recall, that a monoid is a monoidal category \((C, \otimes, 1)\) is a triple \((M, m, e)\), where \(M\) is an object of \(C\), \(m : M \otimes M \to M\) and \(e : 1 \to M\) are arrows such that the diagrams

\[
1 \otimes M \xrightarrow{e \otimes id_M} M \otimes M \xrightarrow{id_M \otimes e} M \otimes 1
\]

\[
(M \otimes M) \otimes M \xrightarrow{\sim} M \otimes (M \otimes M)
\]
Examples of monoids

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- Monoids in the monoidal category of complete join semilattices \((\text{Sup}, \otimes, 2)\) are quantales.
- Monoids in the monoidal category of ordinary monoids \((\text{Mon}, \times, 1)\) are commutative monoids.
Monoids in $\mathbf{Rel}$

So a monoid in the monoidal category $(\mathbf{Rel}, \times, 1)$ consists of

- a set $M$,
- a relation $e : 1 \to M$ and
- a relation $\ast : M \times M \to M$.

such that some diagrams commute.
We call these objects relational monoids
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When dealing with monoids in \textbf{Rel}, one should bear in mind that both $m : A \times A \to A$ and $e : 1 \to A$ are relations, and not mappings. That means, among other things, that

- $e$ is (essentially) a subset of the underlying set, rather than an element;
- it is misleading to write $a \ast b = c$ to express the fact that $(a, b)$ is in the relation $\ast$ with $c$;
- we write $(a, b) \mapsto c$ instead.
Classes of monoids in \textbf{Rel}

- Ordinary monoids are monoids in \textbf{Rel}.

- Hypergroups/hypermonoids are monoids in \textbf{Rel}.

- Partial monoids (including effect algebras and some of their generalizations) are monoids in \textbf{Rel}.

- Small categories are monoids in \textbf{Rel}:
  - elements are arrows,
  - the $e:1 \to M$ is the selection of identity arrows.
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- Small categories are monoids in \textbf{Rel}:
  - elements are arrows,
  - the $e : 1 \to M$ is the selection of identity arrows.
The class of monoids in a monoidal category comes equipped with a standard notion of morphism:

\[
\begin{array}{ccc}
M \times M & \xrightarrow{h \times h} & M' \times M' \\
\downarrow \ast & & \downarrow \ast \\
M & \xrightarrow{h} & M'
\end{array}
\]

However, this notion does not work in the examples we are interested in.
Rel as a 2-category

- A relation $f \subseteq A \times B$ is a set of pairs, so
- every homset $\text{Rel}(A, B)$ is a poset under $\subseteq$.
- That means, that $\text{Rel}$ is enriched in $\text{Pos}$, in other words
- $\text{Rel}$ a (locally posetal/thin) 2-category.
Morphisms of monoids in \textbf{Rel}

There are several meaningful notions of morphisms of monoids in \textbf{Rel}. In this talk, we shall deal with two of them.

\begin{align*}
M \times M & \xrightarrow{h \times h} M' \times M' \\
M & \xrightarrow{h} M' \\
\ast & \\
\downarrow & \\
M & \xrightarrow{h} M'
\end{align*}

Lax morphism

\begin{align*}
(h \circ \ast) & \subseteq (* \circ (h \times h)) \\
h \circ e & \subseteq e
\end{align*}

\begin{align*}
M \times M & \xrightarrow{h \times h} M' \times M' \\
M & \xrightarrow{h} M' \\
\ast & \\
\downarrow & \\
M & \xrightarrow{h} M'
\end{align*}

Oplax morphism

\begin{align*}
(h \circ \ast) & \supseteq (* \circ (h \times h)) \\
e & \subseteq h \circ e
\end{align*}
Category of relational monoids \textbf{RelMon}

By \textbf{Category of relational monoids} we mean a 2-category

- 0-cells are relational monoids,
- 1-cells are lax morphisms of relational monoids,
- 2-cells are the $\subseteq$ of relations, inherited from \textbf{Rel}.
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- The category of small categories is a 1-subcategory of this category.
- The category of effect algebras is a subcategory of this category.
Effect algebras

An effect algebra (Foulis and Bennett [1994], Kôpka and Chovanec [1994], Giuntini and Greuling [1989]) is a partial algebra $(E; +, 0, 1)$ with a binary partial operation $+$ and two nullary operations $0, 1$ such that $+$ is commutative, associative and the following pair of conditions is satisfied:

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a + a'$ exists and $a + a' = 1$.

(E4) If $a + 1$ is defined, then $a = 0$.

The $+$ operation is then cancellative and $0$ is a neutral element.
Why?

- Because \textbf{RelMon} is a 2-category, so it has a lot of structure.
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- Because \textbf{RelMon} is a 2-category, so it has a lot of structure.
- We can take the standard definitions of 2-categorial things from \textbf{RelMon} and examine what they mean for effect algebras.
- We rediscover well-known notions, but now we know where they are coming from.
Easy and nice

Let $E$ be an effect algebra.
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- $\leq : E \to E$ is a left Kan extension of $+$ along the projection $p_1 : E \times E \to E$. 
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- $\leq : E \to E$ is a left Kan extension of $+$ along the projection $p_1 : E \times E \to E$.
- An effect algebra $E$ satisfies the Riesz decomposition property iff $\geq$ is an endomorphism of $E$. 

Easy and nice
Since \textbf{RelMon} is a 2-category, we may speak about adjoint pairs of morphisms in \textbf{RelMon}. Unwinding the definition, it turns out every left adjoint in \textbf{RelMon} is a mapping. Let

\begin{itemize}
  \item $A, B$ be relational monoids,
  \item $f : A \rightarrow B$,
  \item $g : B \rightarrow A$.
\end{itemize}

Then the morphism $f$ is left adjoint to the morphism $g$, if and only if $f$ is a mapping and $g = f^{-1}$. 
Left adjoints in $\textbf{RelMon}$

From this, we obtain a characterization of left adjoints:

**Proposition**

A morphism $f : A \to B$ in $\textbf{RelMon}$ is a left adjoint if and only if $f$ is a mapping and

1. for all $b_1, b_2 \in B$ and $a \in A$ such that $(b_1, b_2) \mapsto f(a)$,
2. there exist $a_1, a_2 \in A$ such that $b_1 = f(a_1)$, $b_2 = f(a_2)$ and $(a_1, a_2) \mapsto a$. 

What if $A$ and $B$ are effect algebras?

**Theorem**

*Let $A, B$ be effect algebras, let $f : A \to B$ be a morphism of effect algebras. Then $f$ is a left adjoint in $\text{RelMon}$ iff*

...
What if $A$ and $B$ are effect algebras?

**Theorem**

Let $A, B$ be effect algebras, let $f : A \to B$ be a morphism of effect algebras. Then $f$ is a left adjoint in $\text{RelMon}$ iff

- $f$ is surjective and
- the equivalence on $A$ induced by $f$ is an effect algebra congruence in the sense of (Gudder and Pulmannová [1998]).
Since \textbf{RelMon} is a 2-category, we may speak about monads in \textbf{RelMon}.

A monad in \textbf{RelMon} on a relational monoid \(A\) can be characterized as a preorder relation \(\leq : A \to A\) such that

\[
\begin{align*}
A \times A & \xrightarrow{\leq \times \leq} A \times A \\
\star & \dashrightarrow \star \\
A & \xrightarrow{\leq} A
\end{align*}
\]

\[
\begin{align*}
1 & \xrightarrow{e} A \\
\leq & \\
e & \dashrightarrow \leq \\
A & \xrightarrow{\leq} A
\end{align*}
\]

commute.
Monads arising from adjunctions in 2-categories

- In every 2-category, an adjoint pair of morphisms gives rise to a monad.
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- In \textbf{Cat}, every monad arises from an adjoint pair (Eilenberg-Moore, Kleisli).
Monads arising from adjunctions in 2-categories

- In every 2-category, an adjoint pair of morphisms gives rise to a monad.
- In $\textbf{Cat}$, every monad arises from an adjoint pair (Eilenberg-Moore, Kleisli).
- But this is not true in every 2-category.
- In particular, in $\textbf{Rel}$ the monads (=preorders) arising from adjunctions can be characterized as equivalence relations.
Monads arising from adjunctions in \( \text{RelMon} \)

If a monad \( \sim : A \to A \) arises from an adjunction, then

- \( \sim \) is an equivalence relation,
- the diagram

\[
\begin{array}{ccc}
A \times A & \xrightarrow{\sim \times \sim} & A \times A \\
\downarrow^{*} & \searrow & \downarrow^{*} \\
A & \xrightarrow{\sim} & A
\end{array}
\]

commutes and

- if \( x \) is a unit of \( A \) and \( x \sim y \), then \( y \) is a unit of \( A \).
If the multiplication is actually a partial operation, we obtain another property of a monad arising from an adjunction:

- If $a_1 \sim b_1$, $a_2 \sim b_2$ and both $a_1 \ast a_2$ and $b_1 \ast b_2$ exist, then $a_1 \ast a_2 \sim b_1 \ast b_2$. 
“Dimension equivalences” on effect algebras

For an effect algebra $E$, we may characterize monads $\sim : E \to E$ arising from adjunctions in $\text{RelMon}$ as follows:

- $\sim$ is an equivalence.
- If $a_1 \sim b_1$, $a_2 \sim b_2$ and both $a_1 + a_2$ and $b_1 + b_2$ exist, then $a_1 + a_2 \sim b_1 + b_2$.
- If $a \sim b_1 + b_2$, then there are $a_1, a_2$ such that $a = a_1 + a_2$, $a_1 \sim b_1$, $a_2 \sim b_2$.

$E/\sim$ is then a partial monoid.
Example

- Take a Boolean algebra \( B \); this is an effect algebra with \(+\) being the disjoint join.
- Introduce a equivalence on \( B \) by the rule

\[
a \sim b \iff [0, a] \simeq [0, b]
\]

Then this is a dimension equivalence.
Let $A$ be an involutive ring with unit, in which

$$x^*x + y^*y = 0 \implies x = y = 0.$$ 

Let $P(A)$ be the set of all self-adjoint idempotents in $A$. For $e, f \in P(A)$, write $e \oplus f = e + f$ iff $ef = 0$, otherwise let $e \oplus f$ be undefined. Then $(P(A); \oplus, 0, 1)$ is an effect algebra.

For $e, f$ in $P(A)$, write $e \sim f$ iff there is $w \in A$ such that $e = w^*w$ and $f = ww^*$.

Then this is a dimension equivalence.
**Theorem**

(Dvurečenskij and Pulmannová [2000]) For every cancellative positive partial abelian monoid $E$ and every dimension equivalence $\sim$ on $P$, $P/\sim$ is a positive partial abelian monoid.
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A new perspective:
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A new perspective:

- $P/\sim$ is the EM-object for the monad $\sim$. 
Let $E$ be an effect algebra. A relation $i : E \to E$ is a comonad in $\text{RelMon}$ iff

$$i = \{(x, x) : x \in I\}$$

where $I$ is an order ideal of $E$. 

Comonads
Let $E$ be an effect algebra. A relation $i : E \to E$ is a comonad in $\text{RelMon}$ iff

\[ i =\{(x, x) : x \in I\} \]

where $I$ is an order ideal of $E$. $I$ is then the EM object for $i$. 
A test space is a pair \((X, \mathcal{T})\), where \(X\) is a set
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- and \(\mathcal{T}\) is a system of subsets of \(X\), called tests,
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- A test space is a pair \((X, \mathcal{T})\), where \(X\) is a set
- and \(\mathcal{T}\) is a system of subsets of \(X\), called tests,
- such that no two tests are comparable.
- A subset of a test is an event.
Two events $a, b$ are said to be orthogonal (in symbols $a \perp b$) if they are disjoint and their union is an event.
Relations on events

- Two events $a, b$ are said to be **orthogonal** (in symbols $a \perp b$) if they are disjoint and their union is an event.
- It is obvious that the set of all events of a test space equipped with the disjoint union of orthogonal events $+$ and $\emptyset$ is a partial commutative monoid.
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- Two events $a, b$ are said to be orthogonal (in symbols $a \perp b$) if they are disjoint and their union is an event.
- It is obvious that the set of all events of a test space equipped with the disjoint union of orthogonal events $+$ and $\emptyset$ is a partial commutative monoid.
- If $a \perp b$ and $a \cup b$ is a test, then they are complements of each other (in symbols $a \text{ co } b$).
A test space is algebraic if \( \text{co} = \text{co} \circ \text{co} \circ \text{co} \).
Algebraic test spaces

- A test space is algebraic if $\co = \co \circ \co \circ \co$.
- For an algebraic test space, the relation $\sim := \co \circ \co$, called perspectivity is an equivalence on events and
- the partial abelian monoid of events, factored by $\sim$ is an orthoalgebra, i.e. and effect algebra with $a \land a' = 0$. 
A characterization of algebraic test spaces

Proposition

Let \((X, \mathcal{T})\) be a test space. Let us write \((A, +, \emptyset)\) for the partial commutative monoid of the events of \((X, \mathcal{T})\). The following are equivalent.

1. \((X, \mathcal{T})\) is algebraic.
2. \(\sim\) is an equivalence relation and an oplax endomorphism of \(A\).
3. \(\sim\) is a preorder and an oplax endomorphism of \(A\).
4. \((A, \tilde{+})\) is associative, where \(\tilde{+} = \circ \sim +\).

Moreover, if \(\sim\) is a preorder and \(\sim\) is an lax and oplax endomorphism of \(A\), then \(A/\sim\) is a Boolean algebra.
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Moreover, if \(~\) is a preorder and \(~\) is an lax and oplax endomorphism of \(A\), then \(A/\sim\) is a Boolean algebra.
Thank you for your attention.


