On two ways of generating topological spaces from Grzegorczyk mereological structures

Rafał Gruszczyński

Department of Logic
Nicolaus Copernicus University in Toruń
Poland

TACL 2015
Ischia, Italy

Outline

1. Two approaches to point-free topology
2. Grzegorczyk semi-lattices
3. Connection relation and connection structures
4. Roeper structures

Outline

1. Two approaches to point-free topology
2. Grzegorczyk semi-lattices
3. Connection relation and connection structures
4. Roeper structures

1st approach – frames and locales

- algebraic properties of the family of open sets

1st approach – frames and locales

- algebraic properties of the family of open sets
- "the study of topology where open-set lattices are taken as the primitive notion" (P. Johnstone *The point of pointless topology*)

1st approach – frames and locales

- algebraic properties of the family of open sets
- "the study of topology where open-set lattices are taken as the primitive notion" (P. Johnstone *The point of pointless topology*)
- how much information about a topological space can be extracted from algebraic properties of its lattice of open sets?

2nd approach – connections structures

- philosophically / foundationally motivated

2nd approach – connections structures

- philosophically / foundationally motivated
- search for an algebraic description of the intended model — the space

2nd approach – connections structures

- philosophically / foundationally motivated
- search for an algebraic description of the intended model — the space
- the primitive notions of this approach are: region, part of relation and connection relation; sometimes additional basic notions are assumed

2nd approach – connections structures

- philosophically / foundationally motivated
- search for an algebraic description of the intended model — the space
- the primitive notions of this approach are: region, part of relation and connection relation; sometimes additional basic notions are assumed
- reverse topology — a common point of the two approaches

The scope of my talk

- two methods of defining points in connections structures

The scope of my talk

- two methods of defining points in connections structures
- with the very basic properties of topological spaces composed of those points

Outline

1. Two approaches to point-free topology
2. Grzegorczyk semi-lattices
3. Connection relation and connection structures
4. Roeper structures

Grzegorczyk semi-lattices (mereology)

A Grzegorczyk semi-lattice is a pair $\langle \mathbb{R}, \sqsubseteq \rangle$ which is a poset such that:

Grzegorczyk semi-lattices (mereology)

A Grzegorczyk semi-lattice is a pair $\langle \mathbb{R}, \sqsubseteq \rangle$ which is a poset such that:

$$x \not\sqsubseteq y \longrightarrow \exists_{z \in M} (z \sqsubseteq x \wedge z \perp y \wedge \forall_{u \in M} (u \sqsubseteq x \wedge u \perp y \longrightarrow u \sqsubseteq z)), \tag{M1}$$

Grzegorczyk semi-lattices (mereology)

A Grzegorczyk semi-lattice is a pair $\langle \mathbb{R}, \sqsubseteq \rangle$ which is a poset such that:

$$x \not\sqsubseteq y \longrightarrow \exists_{z \in M} (z \sqsubseteq x \wedge z \perp y \wedge \forall_{u \in M} (u \sqsubseteq x \wedge u \perp y \longrightarrow u \sqsubseteq z)) \,, \tag{M1}$$

$$\forall_{x,y \in M} \exists_{u \in M}\, u \operatorname{Sup} \{x, y\} \,, \tag{M2}$$

Grzegorczyk semi-lattices (mereology)

A Grzegorczyk semi-lattice is a pair $\langle \mathbb{R}, \sqsubseteq \rangle$ which is a poset such that:

$$\begin{aligned} x \not\sqsubseteq y \longrightarrow \exists_{z \in M} \, (z \sqsubseteq x \wedge z \perp y \wedge \\ \forall_{u \in M} \, (u \sqsubseteq x \wedge u \perp y \longrightarrow u \sqsubseteq z)) \, , \end{aligned} \tag{M1}$$

$$\forall_{x,y \in M} \exists_{u \in M} \, u \; \mathrm{Sup} \; \{x, y\} \, , \tag{M2}$$

$$\forall_{x,y \in M} \, (x \bigcirc y \longrightarrow \exists_{u \in M} \, u \; \mathrm{Inf} \; \{x, y\}) \, , \tag{M3}$$

Grzegorczyk semi-lattices (mereology)

A Grzegorczyk semi-lattice is a pair $\langle \mathbb{R}, \sqsubseteq \rangle$ which is a poset such that:

$$x \not\sqsubseteq y \longrightarrow \exists_{z \in M} \left(z \sqsubseteq x \wedge z \perp y \wedge \forall_{u \in M} \left(u \sqsubseteq x \wedge u \perp y \longrightarrow u \sqsubseteq z \right) \right), \tag{M1}$$

$$\forall_{x,y \in M} \exists_{u \in M}\, u \operatorname{Sup} \{x, y\}, \tag{M2}$$

$$\forall_{x,y \in M} \left(x \bigcirc y \longrightarrow \exists_{u \in M}\, u \operatorname{Inf} \{x, y\} \right), \tag{M3}$$

$$\exists_{x \in M} \forall_{y \in M}\, y \sqsubseteq x. \tag{M4}$$

Outline

1. Two approaches to point-free topology
2. Grzegorczyk semi-lattices
3. Connection relation and connection structures
4. Roeper structures

Quasi-connection structures

A quasi-connection structure is a triple $\langle \mathbb{R}, \sqsubseteq, \mathbb{C} \rangle$ such that $\langle \mathbb{R}, \sqsubseteq \rangle$ is a Grzegorczyk semi-lattice and $\mathbb{C}$ is a binary relation such that:

Quasi-connection structures

A quasi-connection structure is a triple $\langle \mathbb{R}, \sqsubseteq, \mathbb{C} \rangle$ such that $\langle \mathbb{R}, \sqsubseteq \rangle$ is a Grzegorczyk semi-lattice and $\mathbb{C}$ is a binary relation such that:

$$\forall_{x,y \in R} \big(x \sqsubseteq y \longrightarrow x \,\mathbb{C}\, y \big) \,, \tag{C1}$$

Quasi-connection structures

A quasi-connection structure is a triple $\langle \mathbb{R}, \sqsubseteq, \mathbb{C} \rangle$ such that $\langle \mathbb{R}, \sqsubseteq \rangle$ is a Grzegorczyk semi-lattice and $\mathbb{C}$ is a binary relation such that:

$$\forall_{x,y \in R} \big(x \sqsubseteq y \longrightarrow x \,\mathbb{C}\, y \big) \,, \tag{C1}$$

$$\forall_{x,y \in R} \big(x \,\mathbb{C}\, y \longrightarrow y \,\mathbb{C}\, x \big) \,, \tag{C2}$$

Quasi-connection structures

A quasi-connection structure is a triple $\langle \mathbb{R}, \sqsubseteq, \mathbb{C} \rangle$ such that $\langle \mathbb{R}, \sqsubseteq \rangle$ is a Grzegorczyk semi-lattice and $\mathbb{C}$ is a binary relation such that:

$$\forall_{x,y \in R} \big(x \sqsubseteq y \longrightarrow x \,\mathbb{C}\, y \big) , \tag{C1}$$

$$\forall_{x,y \in R} \big(x \,\mathbb{C}\, y \longrightarrow y \,\mathbb{C}\, x \big) , \tag{C2}$$

$$\forall_{x,y,z \in R} \big(x \sqsubseteq y \wedge z \,\mathbb{C}\, x \longrightarrow z \,\mathbb{C}\, y \big) . \tag{C3}$$

Quasi-connection structures

A quasi-connection structure is a triple $\langle \mathbb{R}, \sqsubseteq, \mathbb{C} \rangle$ such that $\langle \mathbb{R}, \sqsubseteq \rangle$ is a Grzegorczyk semi-lattice and $\mathbb{C}$ is a binary relation such that:

$$\forall_{x,y \in R} \big(x \sqsubseteq y \longrightarrow x \,\mathbb{C}\, y\big)\,, \tag{C1}$$

$$\forall_{x,y \in R} \big(x \,\mathbb{C}\, y \longrightarrow y \,\mathbb{C}\, x\big)\,, \tag{C2}$$

$$\forall_{x,y,z \in R} \big(x \sqsubseteq y \wedge z \,\mathbb{C}\, x \longrightarrow z \,\mathbb{C}\, y\big)\,. \tag{C3}$$

$\mathbb{C}$ is called the quasi-connection relation and in case $x \,\mathbb{C}\, y$ we say that x is connected with y.

Separation and non-tangential inclusion of regions

Definition (of separation relation)

$$x \mathrel{)(} y \stackrel{\text{df}}{\longleftrightarrow} \neg x \mathrel{\mathbb{C}} y\,. \qquad (\mathtt{df}\ )()$$

Separation and non-tangential inclusion of regions

Definition (of separation relation)

$$x \mathrel{)(} y \stackrel{\text{df}}{\longleftrightarrow} \neg x \mathbb{C} y \,. \qquad (\mathrm{df}\ \mathrel{)(})$$

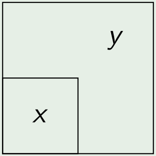
Definition (of non-tangential inclusion)

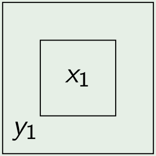
$$\begin{aligned} x \ll y \stackrel{\text{df}}{\longleftrightarrow} & \forall_{z \in \mathbb{R}} (z \mathbb{C} x \longrightarrow z \bigcirc y) \\ \longleftrightarrow & \ y = \mathbf{1} \vee (y \neq \mathbf{1} \wedge x \mathrel{)(} -y) \,. \end{aligned} \qquad (\mathrm{df} \ll)$$

If $x \ll y$ we say that x is non-tangentially included in y.

Example

Region x touches the complement of y, but x_1 is non-tangentially included in y_1, that is $x_1 \ll y_1$.





Auxiliary definitions

Definition

A set of regions X is contracting iff:

$$\forall_{x \in X} \exists_{y \in X}\, y \ll x\,.$$

Auxiliary definitions

Definition

A set of regions X is contracting iff:

$$\forall_{x \in X} \exists_{y \in X}\, y \ll x\,.$$

Definition

$$y \infty X \overset{\text{df}}{\longleftrightarrow} \forall_{y \in X}\, x \,\mathbb{C}\, y \qquad (\mathtt{df}_1 \infty)$$

Auxiliary definitions

Definition

A set of regions X is contracting iff:

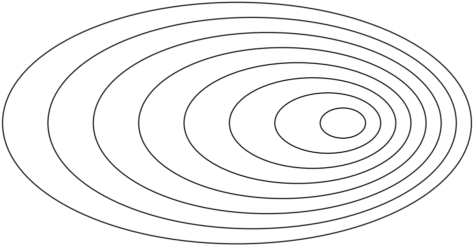
$$\forall_{x \in X} \exists_{y \in X}\, y \ll x\,.$$

Definition

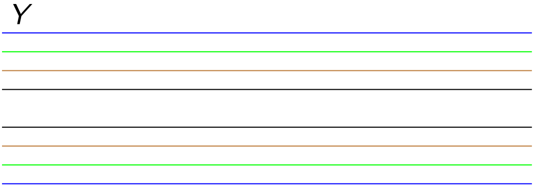
$$y \infty X \overset{\text{df}}{\longleftrightarrow} \forall_{y \in X}\, x \mathbin{\mathbb{C}} y \qquad (\mathtt{df}_1 \infty)$$

$$X \infty Y \overset{\text{df}}{\longleftrightarrow} \forall_{x \in X}\, x \infty Y\,. \qquad (\mathtt{df}_2 \infty)$$

1st intution about points



1st intution about points



Representatives of points

Definition (of a representative of a point)

We say that $X \in \mathcal{P}_+(\mathbb{R})$ is a representative of a point (or a quasi-point) iff X is a contracting set of regions such that:

Representatives of points

Definition (of a representative of a point)

We say that $X \in \mathcal{P}_+(\mathbb{R})$ is a representative of a point (or a quasi-point) iff X is a contracting set of regions such that:

$$\forall_{u,v\in X}(u \neq v \longrightarrow u \ll v \vee v \ll u)\,, \tag{Q1}$$

Representatives of points

Definition (of a representative of a point)

We say that $X \in \mathcal{P}_+(\mathbb{R})$ is a representative of a point (or a quasi-point) iff X is a contracting set of regions such that:

$$\forall_{u,v \in X}(u \neq v \longrightarrow u \ll v \vee v \ll u)\,, \tag{Q1}$$

$$\forall_{u,v \in \mathbb{R}}\big(u \propto X \propto v \longrightarrow u \,\mathbb{C}\, v\big)\,. \tag{Q2}$$

Representatives of points

Definition (of a representative of a point)

We say that $X \in \mathcal{P}_+(\mathbb{R})$ is a representative of a point (or a quasi-point) iff X is a contracting set of regions such that:

$$\forall_{u,v \in X} (u \neq v \longrightarrow u \ll v \vee v \ll u)\,, \tag{Q1}$$

$$\forall_{u,v \in \mathbb{R}} \big(u \infty X \infty v \longrightarrow u \,\mathbb{C}\, v\big)\,. \tag{Q2}$$

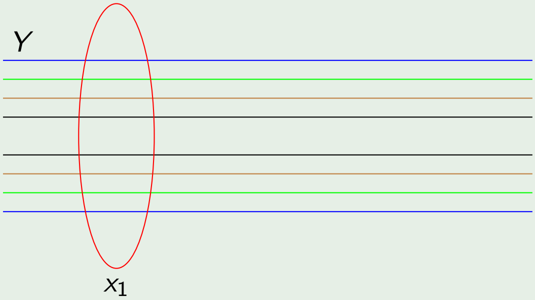
Q — the family of all representatives of points.

Representatives of points

Example

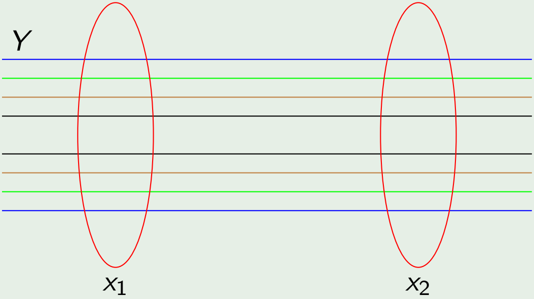
Representatives of points

Example



Representatives of points

Example

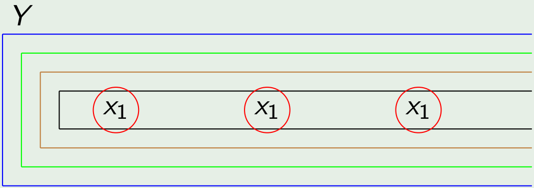


Representatives of points

Example

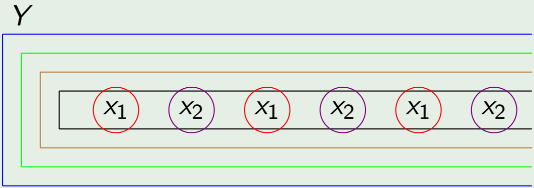
Representatives of points

Example

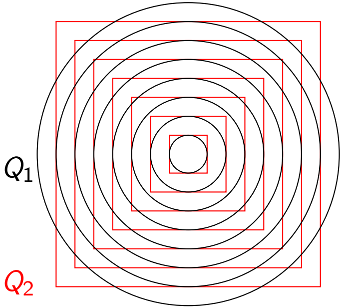


Representatives of points

Example



Definition of a point – short explanation



Q_1 and Q_2 represent the same point.

Points in G-structures

Definition (of a filter)

A filter in a Grzegorczyk semi-lattice $\langle \mathbb{R}, \sqsubseteq \rangle$ is any non-empty set $\mathscr{F} \subseteq \mathbb{R}$ such that

(f1) if $x, y \in \mathscr{F}$, then $x \bigcirc y$ and $x \sqcap y \in \mathscr{F}$,

(f2) if $x \in \mathscr{F}$ and $x \sqsubseteq y$, then $y \in \mathscr{F}$.

Points in G-structures

Definition (of a filter)

A filter in a Grzegorczyk semi-lattice $\langle \mathbb{R}, \sqsubseteq \rangle$ is any non-empty set $\mathscr{F} \subseteq \mathbb{R}$ such that

(f1) if $x, y \in \mathscr{F}$, then $x \bigcirc y$ and $x \sqcap y \in \mathscr{F}$,

(f2) if $x \in \mathscr{F}$ and $x \sqsubseteq y$, then $y \in \mathscr{F}$.

Definition (of a point)

By a point in any G-structure we will mean any filter in $\langle \mathbb{R}, \sqsubseteq \rangle$ generated by a representative of a point. Let us denote the set of all points by 'Π'; for any set β from $\mathcal{P}(\mathbb{R})$:

$$\beta \in \Pi \longleftrightarrow \exists_{Q \in \mathbf{Q}}\, \beta = \{ x \in \mathbb{R} \mid \exists_{y \in Q}\, y \sqsubseteq x \} . \qquad (\mathrm{df}\,\Pi)$$

Existence of points — Grzegorczyk structures

$$x \mathbb{C} y \longrightarrow \exists_{Q \in \mathbf{Q}} \big(x \propto Q \propto y \wedge \exists_{z \in Q} (x \bigcirc y \longrightarrow z \sqsubseteq x \sqcap y) \big) \qquad \texttt{(G)}$$

Existence of points — Grzegorczyk structures

$$x \mathbb{C} y \longrightarrow \exists_{Q \in \mathbf{Q}} \big(x \propto Q \propto y \wedge \exists_{z \in Q} (x \bigcirc y \longrightarrow z \sqsubseteq x \sqcap y)\big) \qquad (\mathtt{G})$$

Any quasi-connection structure $\langle \mathbb{R}, \sqsubseteq, \mathbb{C} \rangle$ satisfying the aforementioned axiom will be called Grzegorczyk structure or G-structure.

Points are maximal contracting filters

Lemma

If $\alpha \in \Pi$, $\mathscr{F}$ *is a contracting filter such that* $\alpha \subseteq \mathscr{F}$, *then* $\alpha = \mathscr{F}$:

$$\Pi \subseteq \mathrm{mcf}(\mathfrak{M}) .$$

Points are maximal contracting filters

Lemma

If $\alpha \in \Pi$, $\mathscr{F}$ *is a contracting filter such that* $\alpha \subseteq \mathscr{F}$, *then* $\alpha = \mathscr{F}$:

$$\Pi \subseteq \mathrm{mcf}(\mathfrak{M})\,.$$

- What about the inverse inclusion: $\mathrm{mcf}(\mathfrak{M}) \subseteq \Pi$?

Points are maximal contracting filters

Lemma

If $\alpha \in \Pi$, $\mathscr{F}$ is a contracting filter such that $\alpha \subseteq \mathscr{F}$, then $\alpha = \mathscr{F}$:

$$\Pi \subseteq \mathrm{mcf}(\mathfrak{M})\,.$$

- What about the inverse inclusion: $\mathrm{mcf}(\mathfrak{M}) \subseteq \Pi$?
- The problem: existence of cofinal reprsentatives of points.

Representation theorems

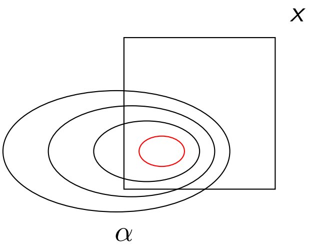
On the set $\mathbb{R}$ we define an operation that ascribes to an arbitrary region x the set of all points α such that $x \in \alpha$.

Representation theorems

On the set $\mathbb{R}$ we define an operation that ascribes to an arbitrary region x the set of all points α such that $x \in \alpha$.

I.e. we define the operation $[\![\mathrm{r}]\!] \colon \mathbb{R} \to \mathcal{P}_{+}(\Pi)$ such that:

$$[\![\mathrm{r}]\!](x) \stackrel{\mathrm{df}}{=} \{\, \alpha \in \Pi \mid x \in \alpha \,\}\,. \tag{df $[\![\mathrm{r}]\!]$}$$



Representation theorems

Lemma

For a given Grzegorczyk structure the set:

$$\{\mathbb{I}\mathbb{r}\mathbb{I}(x) \mid x \in \mathbb{R}\}$$

satisfies conditions of a basis of a topological space.

Theorem

For a given Grzegorczyk structure its topological space $\langle \Pi, \mathscr{O} \rangle$ *is a Hausdorff space.*

Representation theorems

Theorem

If $\mathfrak{M}$ is a Grzegorczyk structure and $\langle \Pi, \mathscr{O} \rangle$ is its topological space, then $[\![\mathrm{r}]\!]$ is an embedding of $\mathfrak{M}$ into $\langle \mathrm{r}\mathscr{O}^+, \subseteq,][\rangle$ where:

$$A \mathrel{][} B \stackrel{\text{df}}{\longleftrightarrow} \mathsf{Cl}\, A \cap \mathsf{Cl}\, B = \emptyset\,. \tag{df][}$$

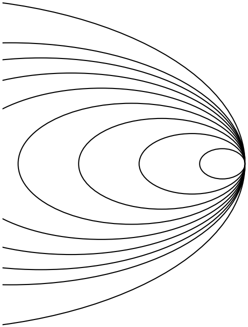
Theorem (A. Grzegorczyk)

If $\mathfrak{M}$ is a complete Grzegorczyk structure and $\langle \Pi, \mathscr{O} \rangle$ is its topological space, then $\mathfrak{M}$ is isomorphic with $\langle \mathrm{r}\mathscr{O}^+, \subseteq,][\rangle$.

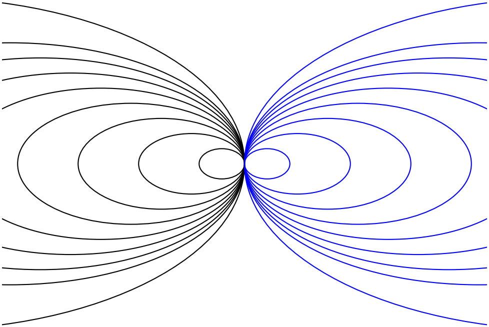
Outline

1. Two approaches to point-free topology
2. Grzegorczyk semi-lattices
3. Connection relation and connection structures
4. Roeper structures

2nd intution about points



2nd intution about points



Definition of a Roeper structure

A Roeper structure is a quadruple $\langle \mathbb{R}, \mathbb{L}, \sqsubseteq, \mathbb{C} \rangle$ such that:

- $\langle \mathbb{R}, \sqsubseteq, \mathbb{C} \rangle$ is a quasi connection structure which satisfies:

Definition of a Roeper structure

A Roeper structure is a quadruple $\langle \mathbb{R}, \mathbb{L}, \sqsubseteq, \mathbb{C} \rangle$ such that:

- $\langle \mathbb{R}, \sqsubseteq, \mathbb{C} \rangle$ is a quasi connection structure which satisfies:

$$x \,\mathbb{C}\, y \sqcup z \longrightarrow x \,\mathbb{C}\, y \vee x \,\mathbb{C}\, z\,, \tag{C4}$$

Definition of a Roeper structure

A Roeper structure is a quadruple $\langle \mathbb{R}, \mathbb{L}, \sqsubseteq, \mathbb{C} \rangle$ such that:

- $\langle \mathbb{R}, \sqsubseteq, \mathbb{C} \rangle$ is a quasi connection structure which satisfies:

$$x \mathbin{\mathbb{C}} y \sqcup z \longrightarrow x \mathbin{\mathbb{C}} y \vee x \mathbin{\mathbb{C}} z\,, \tag{C4}$$

$$\forall_{x \in \mathbb{R}} \exists_{y \in \mathbb{R}}\, y \ll x\,, \tag{$\nexists \mathtt{M}_{\ll}$}$$

Definition of a Roeper structure

A Roeper structure is a quadruple $\langle \mathbb{R}, \mathbb{L}, \sqsubseteq, \mathbb{C} \rangle$ such that:

- $\langle \mathbb{R}, \sqsubseteq, \mathbb{C} \rangle$ is a quasi connection structure which satisfies:

$$x \,\mathbb{C}\, y \sqcup z \longrightarrow x \,\mathbb{C}\, y \vee x \,\mathbb{C}\, z\,, \tag{C4}$$

$$\forall_{x \in \mathbb{R}} \exists_{y \in \mathbb{R}}\, y \ll x\,, \tag{$\nexists \mathtt{M}_{\ll}$}$$

- $\mathbb{L} \subseteq \mathbb{R}$ (the set of limited regions) satisfies the following axioms:

$$x \in \mathbb{L} \wedge y \sqsubseteq x \longrightarrow y \in \mathbb{L}\,, \tag{L1}$$

$$x \in \mathbb{L} \vee y \in \mathbb{L} \longrightarrow x \sqcup y \in \mathbb{L}\,, \tag{L2}$$

Definition of a Roeper structure

A Roeper structure is a quadruple $\langle \mathbb{R}, \mathbb{L}, \sqsubseteq, \mathbb{C} \rangle$ such that:

- $\langle \mathbb{R}, \sqsubseteq, \mathbb{C} \rangle$ is a quasi connection structure which satisfies:

$$x \mathbin{\mathbb{C}} y \sqcup z \longrightarrow x \mathbin{\mathbb{C}} y \vee x \mathbin{\mathbb{C}} z\,, \tag{C4}$$

$$\forall_{x \in \mathbb{R}} \exists_{y \in \mathbb{R}}\, y \ll x\,, \tag{$\nexists \mathtt{M}_{\ll}$}$$

- $\mathbb{L} \subseteq \mathbb{R}$ (the set of limited regions) satisfies the following axioms:

$$x \in \mathbb{L} \wedge y \sqsubseteq x \longrightarrow y \in \mathbb{L}\,, \tag{L1}$$

$$x \in \mathbb{L} \vee y \in \mathbb{L} \longrightarrow x \sqcup y \in \mathbb{L}\,, \tag{L2}$$

$$x \mathbin{\mathbb{C}} y \longrightarrow \exists_{z \in \mathbb{L}} (z \sqsubseteq y \wedge x \mathbin{\mathbb{C}} z)\,, \tag{L3}$$

Definition of a Roeper structure

A Roeper structure is a quadruple $\langle \mathbb{R}, \mathbb{L}, \sqsubseteq, \mathbb{C} \rangle$ such that:

- $\langle \mathbb{R}, \sqsubseteq, \mathbb{C} \rangle$ is a quasi connection structure which satisfies:

$$x \,\mathbb{C}\, y \sqcup z \longrightarrow x \,\mathbb{C}\, y \vee x \,\mathbb{C}\, z\,, \tag{C4}$$

$$\forall_{x \in \mathbb{R}} \exists_{y \in \mathbb{R}}\, y \ll x\,, \tag{$\nexists\mathtt{M}_{\ll}$}$$

- $\mathbb{L} \subseteq \mathbb{R}$ (the set of limited regions) satisfies the following axioms:

$$x \in \mathbb{L} \wedge y \sqsubseteq x \longrightarrow y \in \mathbb{L}\,, \tag{L1}$$

$$x \in \mathbb{L} \vee y \in \mathbb{L} \longrightarrow x \sqcup y \in \mathbb{L}\,, \tag{L2}$$

$$x \,\mathbb{C}\, y \longrightarrow \exists_{z \in \mathbb{L}} (z \sqsubseteq y \wedge x \,\mathbb{C}\, z)\,, \tag{L3}$$

$$x \in \mathbb{L} \wedge x \ll y \longrightarrow \exists_{z \in \mathbb{L}}\, x \ll z \ll y\,. \tag{L4}$$

Limited filters and transitivity

Fact

The relation ∞ is reflexive and symmetrical.

Definition

A filter $\mathscr{F} \subseteq \mathcal{P}(\mathbb{R})$ in a Roeper structure is limited iff it contains at least one limited region.

Theorem

If $\mathscr{U}_1, \mathscr{U}_2$ and $\mathscr{U}_3$ are limited ultrafilters of a Roeper structure, then:

$$\mathscr{U}_1 \infty \mathscr{U}_2 \wedge \mathscr{U}_2 \infty \mathscr{U}_3 \longrightarrow \mathscr{U}_1 \infty \mathscr{U}_3 \,.$$

Limited filters and transitivity

Lemma

If $x \in \mathbb{L}$, $\mathscr{U}$ is an ultrafilters of a Roeper structure, then:

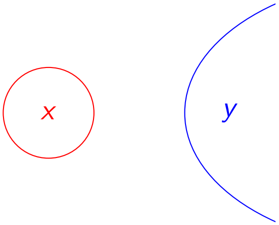
$$x \propto \mathscr{U} \propto y \longrightarrow x \,\mathbb{C}\, y\,.$$

Limited filters and transitivity

Lemma

If $x \in \mathbb{L}$, $\mathscr{U}$ is an ultrafilters of a Roeper structure, then:

$$x \propto \mathscr{U} \propto y \longrightarrow x \, \mathbb{C} \, y \, .$$

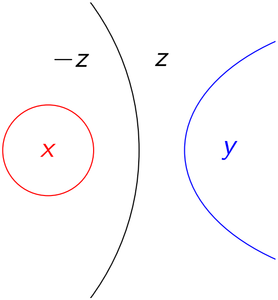


Limited filters and transitivity

Lemma

If $x \in \mathbb{L}$, $\mathscr{U}$ is an ultrafilters of a Roeper structure, then:

$$x \infty \mathscr{U} \infty y \longrightarrow x \, \mathbb{C} \, y \, .$$



Points in Roeper structures

Let $\mathrm{Ult}_\ell(\mathfrak{M})$ be the set of all limited ultrafilter of a Roeper structure $\mathfrak{M}$. Proven that the relation ∞ is an equivalence relation there are two ways of defining points:

- as equivalence classes of ∞ in the set $\mathrm{Ult}_\ell(\mathfrak{M})$:

$$\Pi_1 \stackrel{\mathrm{df}}{=} \{[\mathscr{U}]_\infty \mid \mathscr{U} \in \mathrm{Ult}_\ell(\mathfrak{M})\} \tag{$\mathtt{df}_1\,\Pi$}$$

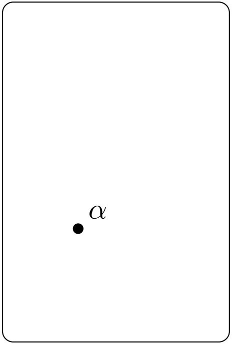
- as interesections of those classes, i.e.

$$\Pi_2 \stackrel{\mathrm{df}}{=} \left\{\bigcap[\mathscr{U}]_\infty \mid \mathscr{U} \in \mathrm{Ult}_\ell(\mathfrak{M})\right\}. \tag{$\mathtt{df}_2\,\Pi$}$$

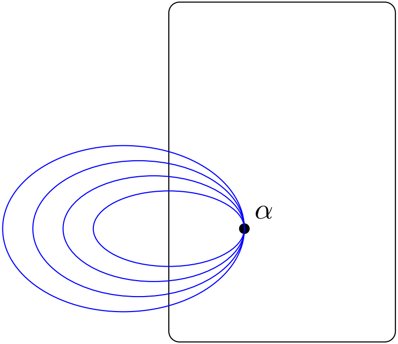
- there is a one-to-one correspondence between them, i.e.:

$$\bigcap[\mathscr{U}_1]_\infty = \bigcap[\mathscr{U}_2]_\infty \longrightarrow \mathscr{U}_1 \infty \mathscr{U}_2 .$$

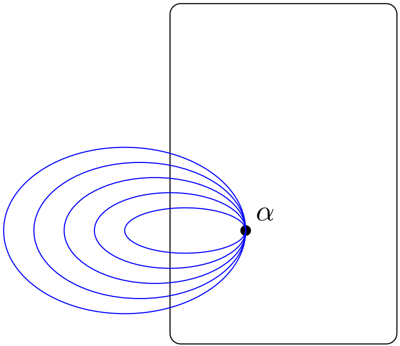
General idea behind points as intersections



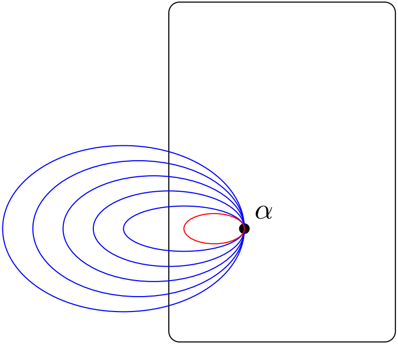
General idea behind points as intersections



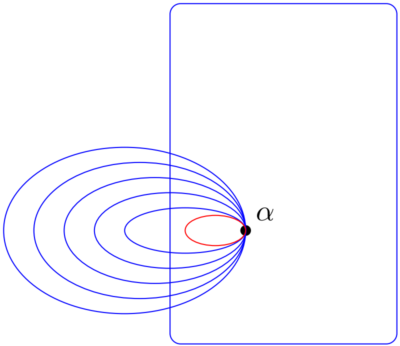
General idea behind points as intersections



General idea behind points as intersections



General idea behind points as intersections



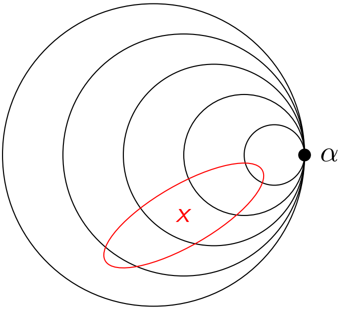
In this way we identify the point α with all regions in whose «interior» α is located.

Topology in Roeper structures

- Similarly as in case of G-structures we may introduce a topology $\mathscr{O}$ on Π, about which one may prove that it is Hausdorff and locally compact.
- For limited region x, the set:

$$\left\{\bigcap[\mathscr{U}]_\infty \mid x \in \bigcup[\mathscr{U}]_\infty\right\}$$

is a compact set in $\langle X, \mathscr{O}\rangle$.



Comparing Grzegorczyk and Roeper definitions

- From point of view of comparison of both definitions, the latter (points as $\bigcap[\mathscr{U}]_\infty$) is more useful.

Comparing Grzegorczyk and Roeper definitions

- From point of view of comparison of both definitions, the latter (points as $\bigcap[\mathscr{U}]_\infty$) is more useful.
- $\mathrm{mcf}_{\mathbb{L}}(\mathfrak{M}) = \{\bigcap[\mathscr{U}]_\infty \mid \mathscr{U} \in \mathrm{Ult}_{\mathbb{L}}(\mathfrak{M})\}$

Comparing Grzegorczyk and Roeper definitions

- From point of view of comparison of both definitions, the latter (points as $\bigcap[\mathscr{U}]_\infty$) is more useful.
- $\mathrm{mcf}_{\mathbb{L}}(\mathfrak{M}) = \{\bigcap[\mathscr{U}]_\infty \mid \mathscr{U} \in \mathrm{Ult}_{\mathbb{L}}(\mathfrak{M})\}$
- It seems that points in the sense of Roeper should satisfy Grzegorczyk's definition — the problem is that we have to show that for every $\bigcap[\mathscr{U}]_\infty$ there exists a representative of a point in the sense of Grzegorczyk which generates $\bigcap[\mathscr{U}]_\infty$.

Comparing Grzegorczyk and Roeper definitions

- From point of view of comparison of both definitions, the latter (points as $\bigcap[\mathscr{U}]_\infty$) is more useful.
- $\mathrm{mcf}_{\mathbb{L}}(\mathfrak{M}) = \{\bigcap[\mathscr{U}]_\infty \mid \mathscr{U} \in \mathrm{Ult}_{\mathbb{L}}(\mathfrak{M})\}$
- It seems that points in the sense of Roeper should satisfy Grzegorczyk's definition — the problem is that we have to show that for every $\bigcap[\mathscr{U}]_\infty$ there exists a representative of a point in the sense of Grzegorczyk which generates $\bigcap[\mathscr{U}]_\infty$.
- If $\Pi_{\mathtt{G}} = \mathrm{mcf}(\mathfrak{M})$, then $\Pi_{\mathtt{R}} \subseteq \Pi_{\mathtt{G}}$.

Comparing Grzegorczyk and Roeper definitions

- From point of view of comparison of both definitions, the latter (points as $\bigcap[\mathscr{U}]_\infty$) is more useful.
- $\mathrm{mcf}_{\mathbb{L}}(\mathfrak{M}) = \{\bigcap[\mathscr{U}]_\infty \mid \mathscr{U} \in \mathrm{Ult}_{\mathbb{L}}(\mathfrak{M})\}$
- It seems that points in the sense of Roeper should satisfy Grzegorczyk's definition — the problem is that we have to show that for every $\bigcap[\mathscr{U}]_\infty$ there exists a representative of a point in the sense of Grzegorczyk which generates $\bigcap[\mathscr{U}]_\infty$.
- If $\Pi_{\mathtt{G}} = \mathrm{mcf}(\mathfrak{M})$, then $\Pi_{\mathtt{R}} \subseteq \Pi_{\mathtt{G}}$.
- The other direction from Grzegorczyk's points to Reoeper's is harder since it involves the problem of capturing limited regions in the theory of the Polish logician.

Thank you for your attention.