Representation of Free Finitely Generated Weak Nilpotent Minimum Algebras

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Based on a joint work with:
Stefano Aguzzoli (UNIMI) and Simone Bova (TU Wien).

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- Monoidal T-norms based Logic;
- Weak Negations Functions, WNM Algebras and Chains;
- Representation of Free *n*-generated WNM algebras;
- Applications.

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A **T-norm** is a binary operation $\cdot : [0, 1]^2 \to [0, 1]$ that satisfies:

1. $x \cdot y = y \cdot x$
2. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
3. $x \leq y$ then $x \cdot z \leq y \cdot z$
4. $x \cdot 1 = x$

for all $x, y, z, \in [0, 1]$.

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Residuum: $\Rightarrow_\cdot : [0,1]^2 \to [0,1]$.
Adjunction property:
$(x \cdot z) \leq y$ iff $z \leq (x \Rightarrow y)$
that is $x \Rightarrow y = max\{z | x \cdot z \leq y\}$,
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Monoidal T-norm based Logic (MTL)(Esteva and Godo): Many-Valued logic of all left-continuous t-norms and their residua (Montagna and Jenei).

A *commutative integral bounded residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \rightarrow, \bot, \top)$ of type $(2, 2, 2, 2, 0, 0)$ such that
$(A, \wedge, \vee, \bot, \top)$ is a bounded lattice,
$(A, \cdot, \top)$ is a commutative monoid,
and the *residuation* equivalence, $x \cdot y \leq z$ if and only if $x \leq y \rightarrow z$, holds.

An **MTL algebra** $\mathbf{A} = (A, \wedge, \vee, \cdot, \rightarrow, \bot, \top)$ is a commutative integral bounded residuated lattice satisfying the **prelinearity** equation, $(x \rightarrow y) \vee (y \rightarrow x) = \top$.

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The class of unary operation $' \colon [0, 1] \rightarrow [0, 1]$ arising as negation operations of MTL algebras over $[0, 1]$ coincides with the class of weak negation operations.

A **weak negation** is a unary operations $'\colon [0,1] \to [0,1]$ such that, for all $a, b \in [0,1]$:

$$0' = 1; \qquad a \leq b \text{ implies } b' \leq a'; \qquad \text{and, } a \leq a''.$$

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Given a weak negation $': [0,1] \to [0,1]$, it is possible to define a t-norm as follows, for all $x, y \in [0,1]$:

$$x \cdot y = \begin{cases} 0 & \text{if } x \leq y', \\ x \wedge y & \text{otherwise.} \end{cases}$$

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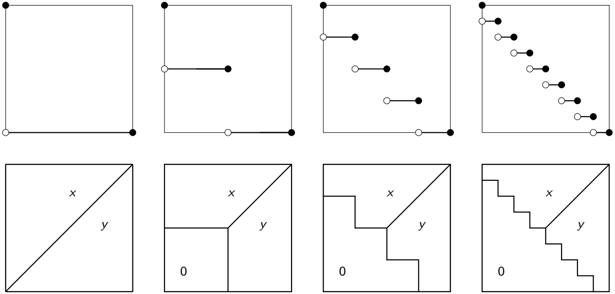
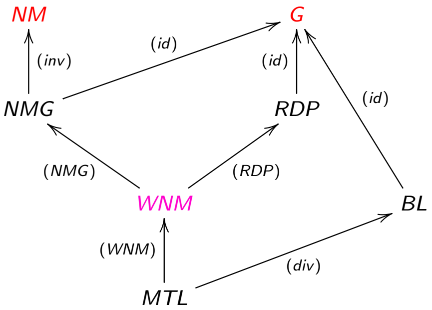


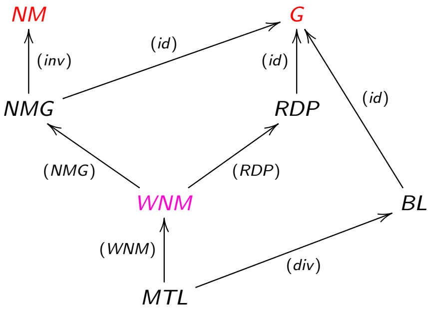
The first four members of the family of weak negations $\{f_n \mid n = 0, 1, 2, \ldots\}$ and their induce t-norms.

f_n is the step function over $[0,1]$ that maps 0 to 1, and $((i-1)/n, i/n)$ to $(n-i)/n$ for $i = 1, 2, \ldots, n$, so that f_n has 2^n discontinuities.


NM
G
(inv)
(id)
(id)
NMG
RDP
(id)
(NMG)
(RDP)
WNM
BL
(WNM)
(div)
MTL

A **WNM Algebra** is a MTL algebra satisfying the *Weak Nilpotent Minimum* equations:

$$\neg(x \cdot y) \vee ((x \wedge y) \to (x \cdot y)) = \top.$$



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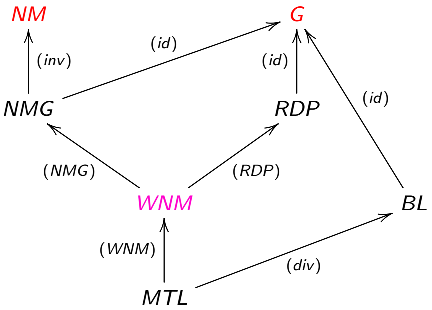

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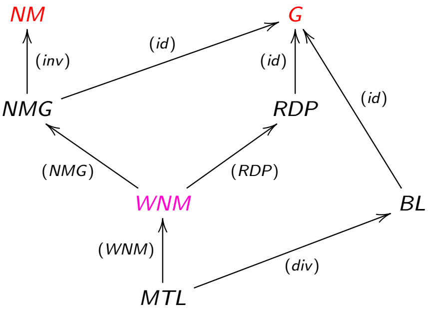
A **NM Algebra** is an *involutive* WNM algebra, that is, a WNM algebra satisfying:

$$\neg\neg x = x.$$



1. The *subdirectly irreducible* members of the variety $\mathbb{V}(MTL)$ are totally ordered (Esteva and Godo);
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$\Rightarrow$ The finitely generated free algebras in subvarieties of $\mathbb{V}(WNM)$ are finite.

Given a set of generators $x_1, \ldots, x_n$,

the WNM algebra $\mathbf{F}_n$ freely generated by $x_i^{\mathbf{F}_n} = (x_i^{\mathbf{C}_1}, \ldots, x_i^{\mathbf{C}_m})$ for $i = 1, \ldots, n$, is isomorphic to the subalgebra $\mathbf{A}$ of $\mathbf{C}_1 \times \cdots \times \mathbf{C}_m$ generated by $x_i^{\mathbf{A}} = (x_i^{\mathbf{C}_1}, \ldots, x_i^{\mathbf{C}_m})$ for $i = 1, \ldots, n$.

The variety $\mathbb{V}(WNM)$ is generated by WNM chains, and for all WNM chains $\mathbf{C}$, the operations $\cdot^{\mathbf{C}}$ and $\rightarrow^{\mathbf{C}}$ are uniquely determined by the lattice and negation operations, as follows (for all $a, b \in C$):

$$a \cdot^{\mathbf{C}} b = \begin{cases} 0^{\mathbf{C}} & \text{if } a \leq^{\mathbf{C}} b'^{\mathbf{C}}, \\ a \wedge^{\mathbf{C}} b & \text{otherwise;} \end{cases}$$

$$a \rightarrow^{\mathbf{C}} b = \begin{cases} 1^{\mathbf{C}} & \text{if } a \leq^{\mathbf{C}} b, \\ a'^{\mathbf{C}} \vee^{\mathbf{C}} b & \text{otherwise.} \end{cases}$$

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Proposition (Esteva, Noguera, Gispert)

For all WNM chains $\mathbf{C}$:

$$\begin{aligned} x \leq x'' = \bigwedge\{z \in C \mid x \leq z, z = z''\}, \\ x = x^2 \text{ iff } x' < x \text{ or } x = 0, \\ x \leq y \text{ implies } y' \leq x', \\ x' < x \text{ and } y' < y \text{ implies } x' < y, \\ x \leq x' \text{ and } y' < y \text{ implies } x \leq y, \\ x' < x \text{ and } y \leq y' \text{ implies } x' < y'. \end{aligned}$$

Let $\mathcal{C}_n = \{\mathbf{C}_1, \ldots, \mathbf{C}_m\}$ be the set of (pairwise non-isomorphic) subdirectly irreducible n-generated WNM algebras.

Let $\mathbf{C}$ be a WNM chain generated by $x_1^{\mathbf{C}}, \ldots, x_n^{\mathbf{C}} \in C$. Then

$$\mathrm{bk}(\mathbf{C}) = B_1 < \cdots < B_k.$$

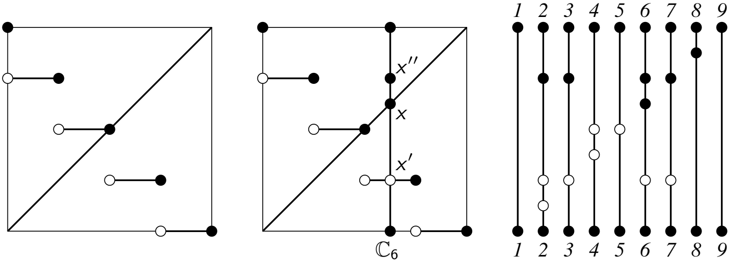
is the n-generated WNM chain such that:

1. the blocks $B_1, \ldots, B_k$ form a partition of $\{0, 1, x_i, x_i', x_i'' \mid i = 1, \ldots, n\}$;
2. the generator $x_i^{\mathrm{bk}(\mathbf{C})}$ is the block containing x_i for $i = 1, \ldots, n$;
3. $x, y \in B_j$ iff $x = y$ for $j = 1, \ldots, k$;
4. $B_j < B_{j+1}$ iff $x < y$, where $x \in B_j$, $y \in B_{j+1}$, $j = 1, \ldots, k-1$;
5. $B_j' = B_l$ iff $x' = y$, where $x \in B_j$, $y \in B_l$, $j = 1, \ldots, k$.

$$\mathrm{bk}(\mathbf{C}_{15}) = 0 < xx'' < y < y'y'' < x' < 1$$

$$\mathrm{bk}(\mathbf{C}_6) = 0 < x < x'' < yy'y'' < x' < 1$$

The class $\mathcal{C}_1$ of singly generated WNM chains $\mathcal{C}_1 = \{\mathbf{C}_i \mid i = 1, \ldots, 9\}$ is:



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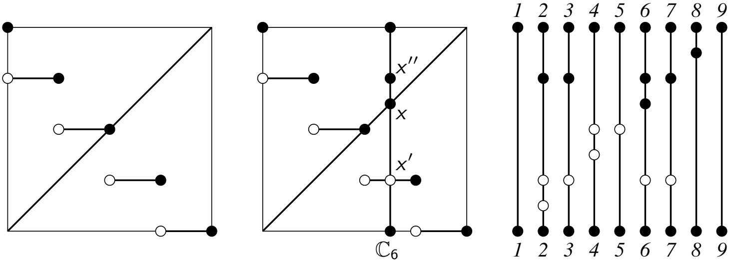


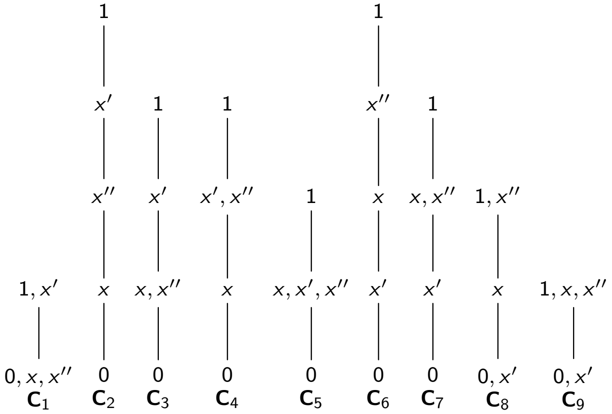
$\mathrm{bk}(\mathbf{C}_1) = 0x_1x_1'' < x_1'1$, $\mathrm{bk}(\mathbf{C}_2) = 0 < x_1 < x_1'' < x_1' < 1$,

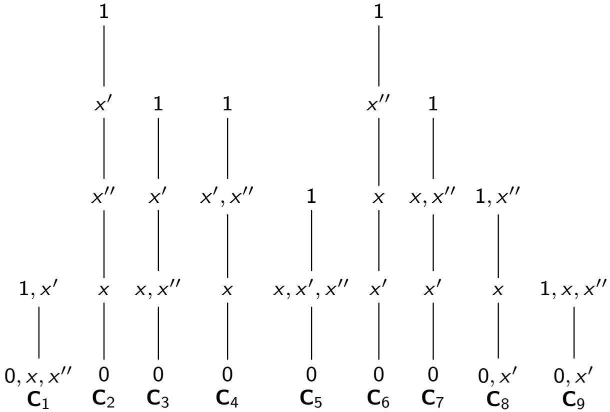
$\mathrm{bk}(\mathbf{C}_3) = 0 < x_1x_1'' < x_1' < 1$, $\mathrm{bk}(\mathbf{C}_4) = 0 < x_1 < x_1'x_1'' < 1$,

$\mathrm{bk}(\mathbf{C}_5) = 0 < x_1x_1'x_1'' < 1$, $\mathrm{bk}(\mathbf{C}_6) = 0 < x_1' < x_1 < x_1'' < 1$,

$\mathrm{bk}(\mathbf{C}_7) = 0 < x_1' < x_1x_1'' < 1$, $\mathrm{bk}(\mathbf{C}_8) = 0x_1' < x_1 < x_1''1$, $\mathrm{bk}(\mathbf{C}_9) = 0x_1' < x_1x_1''1$,

Blockwise Representation

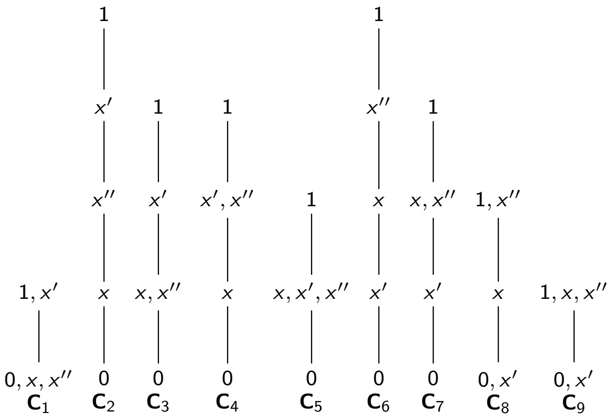




Let $\mathbf{C} \in \mathcal{C}_n$. For $i = 1, \ldots, n$, the **orbit** of x_i in $\mathbf{C}$ is the subalgebra of $\mathbf{C}$ generated by $x_i^{\mathbf{C}}$. We define $\mathrm{orbit}(\mathbf{C}, 0) := 1$, $\mathrm{orbit}(\mathbf{C}, 1) := 9$, and for $i = 1, \ldots, n$,

$$\mathrm{orbit}(\mathbf{C}, x_i) = \mathrm{orbit}(\mathbf{C}, x_i') = \mathrm{orbit}(\mathbf{C}, x_i'') := j,$$

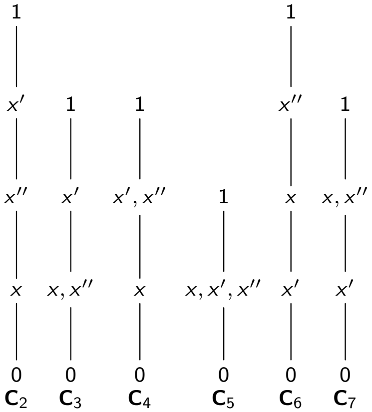
iff the orbit of x_i in $\mathbf{C}$ is isomorphic to $\mathbf{C}_j \in \mathcal{C}_1$, where $j \in \{1, \ldots, 9\}$.



$$\mathcal{K}_n \subseteq \mathcal{C}_n$$

be such that $\mathbf{C} \in \mathcal{K}_n$ iff $\mathbf{C} \in \mathcal{C}_n$ and there does not exist $\mathbf{D} \in \mathcal{C}_n$ and a congruence $\equiv$ on $\mathbf{D}$ above the identity such that $\mathbf{C} = \mathbf{D}/\equiv$.

$$\mathbf{C} \in \mathcal{K}_n \text{ iff } \mathrm{orbit}(\mathbf{C}, x_i) \in \{2, 3, \ldots, 7\} \text{ for all } i = 1, \ldots, n$$



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Let $\mathbf{D} \in \mathcal{K}_n, i = 1, \ldots, n$. We write,
$D_0 := \{0\}$,

$D_1 := \{x_i, x_i'' \mid \text{orbit}(\mathbf{D}, x_i) \in \{2, 3\}\}$
$\cup\{x_i' \mid \text{orbit}(\mathbf{D}, x_i) \in \{6, 7\}\}$,

$D_2 := \{x_i \mid \text{orbit}(\mathbf{D}, x_i) = 4\}$,

$D_3 := \{x_i', x_i'' \mid \text{orbit}(\mathbf{D}, x_i) = 4\}$
$\cup\{x_i, x_i', x_i'' \mid \text{orbit}(\mathbf{D}, x_i) = 5\}$,

$D_4 := \{x_i' \mid \text{orbit}(\mathbf{D}, x_i) \in \{2, 3\}\}$
$\cup\{x_i, x_i'' \mid \text{orbit}(\mathbf{D}, x_i) \in \{6, 7\}\}$,

$D_5 := \{1\}$.

$$l_{\mathbf{D}} = \bigwedge_{y \in D_4 \cup D_5} y, \quad g_{\mathbf{D}} = \bigvee_{y \in \{\bigcup_{j \in [3]} D_j\}} y.$$

$l_{\mathbf{D}}$ is the least element $y \in D$ such that $y' < y$,
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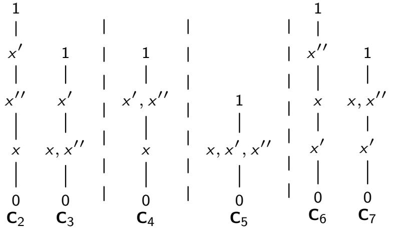

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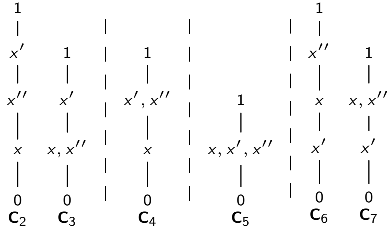
$\mathbf{C}$ and $\mathbf{D}$ in $\mathcal{K}_n$ have the same **signature** (in symbols, $\mathbf{C} \sim \mathbf{D}$) iff:

(S_1) $C_i = D_i$ for $i = 1, 2, 3, 4$;

(S_2) $x \diamond_{\mathbf{C}} y$ iff $x \diamond_{\mathbf{D}} y$ for all $x, y \in C_2, \diamond \in \{<, =\}$.



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$g_{\mathbf{D}}$ is the greatest element $y \in D$ such that $y \leq y'$.

$$l_{\mathbf{D}} \prec g_{\mathbf{D}}$$

C and **D** in $\mathcal{K}_n$ have the same **signature** (in symbols, $\mathbf{C} \sim \mathbf{D}$) iff:

(S_1) $C_i = D_i$ for $i = 1,2,3,4$;

(S_2) $x \diamond_{\mathbf{C}} y$ iff $x \diamond_{\mathbf{D}} y$ for all $x, y \in C_2, \diamond \in \{<, =\}$.

An **infix** of **C** is an interval I in $\mathrm{bk}(\mathbf{C})$ such that:

(I_1) There exists $x \in I$ such that $x = g_{\mathbf{C}}$ or $x = l_{\mathbf{C}}$.

Let $\mathbf{C} \sim \mathbf{D}$ in $\mathcal{K}_n$. Then, $\mathrm{infix}(\mathbf{C}, \mathbf{D})$ is the greatest common infix I of **C** and **D** such that:

(I_2) $x_i \in I$ and $x_i', x_i'' \notin I$, or $x_i, x_i', x_i'' \in I$ for all $i = 1, \dots, n$.

$\mathrm{bk}(\mathbf{C}_5) = 0 < x < x'' < y < y'y'' < x' < 1$;
$\mathrm{bk}(\mathbf{C}_{15}) = 0 < xx'' < y < y'y'' < x' < 1$;

$\mathrm{bk}(\mathbf{C}_6) = 0 < x < x'' < yy'y'' < x' < 1$;
$\mathrm{bk}(\mathbf{C}_{16}) = 0 < xx'' < yy'y'' < x' < 1$;

Theorem

Let t be a term, and $\mathbf{C} \sim \mathbf{D}$ in $\mathcal{K}_n$.

(*i*) $t' < t$ *in* $\mathbf{C}$ *iff* $t' < t$ *in* $\mathbf{D}$.

(*ii*) *For all* $x \in \mathrm{infix}(\mathbf{C}, \mathbf{D})$, $t = x$ *in* $\mathbf{C}$ *iff* $t = x$ *in* $\mathbf{D}$.

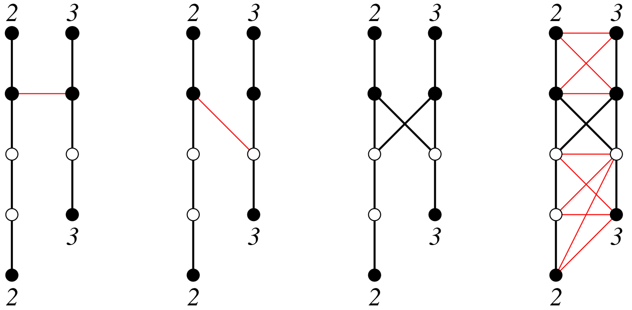
Theorem

Let t be a term, and $\mathbf{C} \sim \mathbf{D}$ in $\mathcal{K}_n$.

(*i*) *$t' < t$ in $\mathbf{C}$ iff $t' < t$ in $\mathbf{D}$.*

(*ii*) *For all $x \in \mathrm{infix}(\mathbf{C}, \mathbf{D})$, $t = x$ in $\mathbf{C}$ iff $t = x$ in $\mathbf{D}$.*

$$\mathrm{bk}(\mathbf{C}_2) = 0 < x_1 < x_1'' < x_1' < 1,$$
$$\mathrm{bk}(\mathbf{C}_3) = 0 < x_1 x_1'' < x_1' < 1.$$

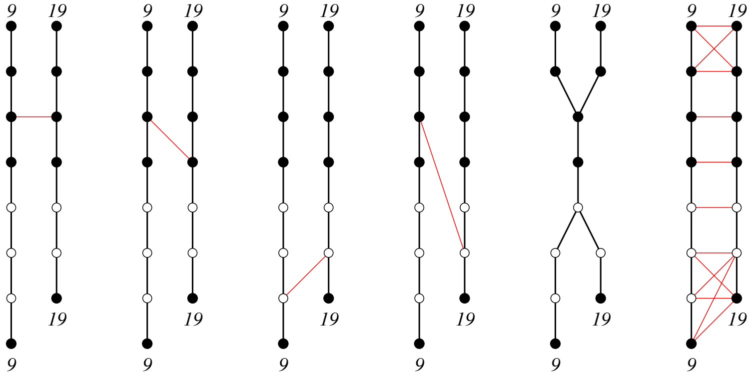


Theorem

Let t be a term, and $\mathbf{C} \sim \mathbf{D}$ in $\mathcal{K}_n$.

- (i) *$t' < t$ in $\mathbf{C}$ iff $t' < t$ in $\mathbf{D}$.*
- (ii) *For all $x \in \mathrm{infix}(\mathbf{C}, \mathbf{D})$, $t = x$ in $\mathbf{C}$ iff $t = x$ in $\mathbf{D}$.*

$$\mathrm{bk}(\mathbf{C}_9) = 0 < x < x'' < y' < y < y'' < x' < 1,$$
$$\mathrm{bk}(\mathbf{C}_{19}) = 0 < xx'' < y' < y < y'' < x' < 1.$$



Theorem

Let t be a term, and $\mathbf{C} \sim \mathbf{D}$ in $\mathcal{K}_n$.

(*i*) $t' < t$ *in* $\mathbf{C}$ *iff* $t' < t$ *in* $\mathbf{D}$.

(*ii*) *For all* $x \in \mathrm{infix}(\mathbf{C}, \mathbf{D})$, $t = x$ *in* $\mathbf{C}$ *iff* $t = x$ *in* $\mathbf{D}$.

$$A \subseteq \prod_{\mathbf{C} \in \mathcal{K}_n} C$$

be such that $a \in A$ iff, for all $\mathbf{C} \sim \mathbf{D}$ in $\mathcal{K}_n$:

(*i*) $\pi_{\mathbf{C}}(a)' < \pi_{\mathbf{C}}(a)$ iff $\pi_{\mathbf{D}}(a)' < \pi_{\mathbf{D}}(a)$;

(*ii*) If $x \in \mathrm{infix}(\mathbf{C}, \mathbf{D})$, then $\pi_{\mathbf{C}}(a) = x$ iff $\pi_{\mathbf{D}}(a) = x$.

Theorem

Let t be a term, and $\mathbf{C} \sim \mathbf{D}$ in $\mathcal{K}_n$.

(i) $t' < t$ in $\mathbf{C}$ iff $t' < t$ in $\mathbf{D}$.

(ii) For all $x \in \mathrm{infix}(\mathbf{C}, \mathbf{D})$, $t = x$ in $\mathbf{C}$ iff $t = x$ in $\mathbf{D}$.

$$A \subseteq \prod_{\mathbf{C} \in \mathcal{K}_n} C$$

be such that $a \in A$ iff, for all $\mathbf{C} \sim \mathbf{D}$ in $\mathcal{K}_n$:

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(ii) If $x \in \mathrm{infix}(\mathbf{C}, \mathbf{D})$, then $\pi_{\mathbf{C}}(a) = x$ iff $\pi_{\mathbf{D}}(a) = x$.

$\mathbf{A} = (A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \cdot^{\mathbf{A}}, \to^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}})$ where:

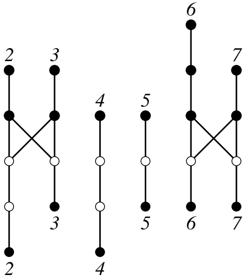
$0^{\mathbf{A}}$ is the least antichain in A;

$1^{\mathbf{A}}$ is the greatest antichain in A;

for $\circ^{\mathbf{A}} \in \{\wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \cdot^{\mathbf{A}}, \to^{\mathbf{A}}\}$ and $a, b \in A$, $\circ^{\mathbf{A}}$ is defined *chainwise*, that is $a \circ^{\mathbf{A}} b = a_C \circ b_C$, for every $\mathbf{C} \in \mathcal{K}_n$.

Theorem (Free Algebras)

The algebra $\mathbf{A}$ is isomorphic to the free n-generated WNM algebra $\mathbf{F}_n$.



$$|\mathbf{F}_1| = 1200$$

$$\mathrm{poset}(n) = \sum_{\substack{a,b,c\in\{0\}\cup\mathbb{N},\\ a+b+c=n}} \binom{n}{a,\, b,\, c} \mathbf{X}_{a,b,c},$$

$$\mathbf{X}_{a,b,c} = \begin{cases} \mathbf{Y}^{d}_{a,c} \oplus \mathbf{Z}_{a,c} & \text{if } b = 0,\\ \sum_{i=1}^{b} i! {b \brace i} \left(\left(\mathbf{Y}^{d}_{a,c} \oplus \mathbf{i} \oplus \mathbf{Z}_{a,c}\right) + \left(\mathbf{Y}^{d}_{a,c} \oplus \mathbf{i} \oplus \mathbf{1} \oplus \mathbf{Z}_{a,c}\right)\right) & \text{otherwise;} \end{cases}$$

$$\mathbf{Y}_{a,c} = \begin{cases} \mathbf{1} & \text{if } a = c = 0,\\ \sum_{i=0}^{a-1}\sum_{j=0}^{c-1} \binom{a}{i}\binom{c}{j} \left(\mathbf{S}_{i,j} + \mathbf{S}^{-}_{i,j}\right) + \\ \sum_{j=0}^{c-1} \binom{c}{j} \left(\mathbf{S}_{a,j} + \mathbf{S}^{+}_{a,j}\right) + \\ \sum_{i=0}^{a-1} \binom{a}{i} \left(\mathbf{S}_{i,c} + \mathbf{S}^{-}_{i,c}\right) & \text{otherwise;} \end{cases}$$

$$\mathbf{S}_{i,j} = \mathbf{1} \oplus \mathbf{Y}_{i,j},$$

$$\mathbf{S}^{+}_{i,j} = \begin{cases} \mathbf{2} & \text{if } i = j = 0,\\ \mathbf{S}_{i,j} + \mathbf{Y}_{i,j} & \text{otherwise;} \end{cases}$$

$$\mathbf{S}^{-}_{i,j} = (\mathbf{1} \oplus \mathbf{S}_{i,j}) + \sum_{k=0}^{i-1} \binom{i}{k} \left((\mathbf{1} \oplus \mathbf{S}_{k,j}) + (\mathbf{1} \oplus \mathbf{S}^{-}_{k,j})\right).$$

$$\mathbf{Z}_{a,c} = \begin{cases} \mathbf{1} & \text{if } a = c = 0,\\ \sum_{i=0}^{a-1}\sum_{j=0}^{c-1} \binom{a}{i}\binom{c}{j} \left(\mathbf{T}_{i,j} + \mathbf{T}^{-}_{i,j}\right) + \\ \sum_{j=0}^{c-1} \binom{c}{j} \left(\mathbf{T}_{a,j} + \mathbf{T}^{+}_{a,j}\right) + \\ \sum_{i=0}^{a-1} \binom{a}{i} \left(\mathbf{T}_{i,c} + \mathbf{T}^{-}_{i,c}\right) & \text{otherwise;} \end{cases}$$

$$\mathbf{T}_{i,j} = \mathbf{1} \oplus \mathbf{Z}_{i,j},$$

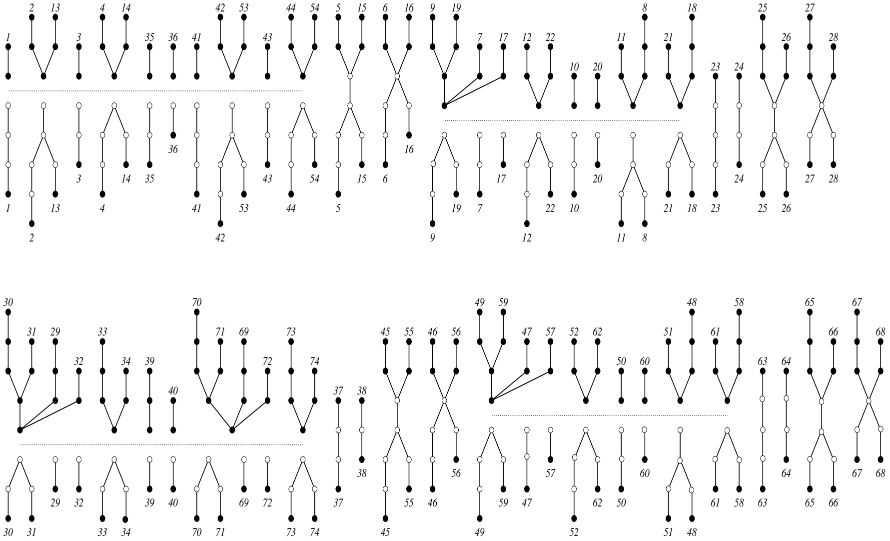
$$\mathbf{T}^{+}_{i,j} = \begin{cases} \mathbf{3} & \text{if } i = j = 0,\\ \mathbf{1} \oplus (\mathbf{T}_{i,j} + \mathbf{Z}_{i,j}) & \text{otherwise;} \end{cases}$$

$$\mathbf{T}^{-}_{i,j} = \mathbf{T}_{i,j} + \sum_{k=0}^{i-1} \binom{i}{k} \left(\mathbf{T}_{k,j} + \mathbf{T}^{-}_{k,j}\right).$$

$\mathbf{P} + \mathbf{Q} := (P \cup Q, \leq^{\mathbf{P}+\mathbf{Q}})$ where $p \leq^{\mathbf{P}+\mathbf{Q}} q$ if and only if either $p, q \in P$ and $p \leq^{\mathbf{P}} q$, or $p, q \in Q$ and $p \leq^{\mathbf{Q}} q$.

$\mathbf{P} \oplus \mathbf{Q} := (P \cup Q, \leq^{\mathbf{P}\oplus\mathbf{Q}})$ where $p \leq^{\mathbf{P}\oplus\mathbf{Q}} q$ if and only if either $p, q \in P$ and $p \leq^{\mathbf{P}} q$, or $p, q \in Q$ and $p \leq^{\mathbf{Q}} q$, or $p \in P$ and $q \in Q$.

Free Algebras



$$|\mathbf{F}_2| = 1.2495275042442405\cdot 10^{32} = 124,952,750,424,424,055,667,787,038,720,000$$

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- Normal Forms;
- Recurrence Formulas;
- Interpolation Properties;
- Unification;
- Spectral Duality.

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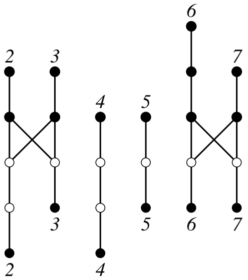

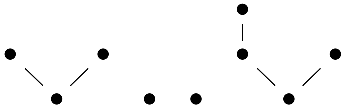

- Normal Forms;
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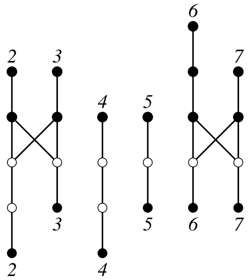

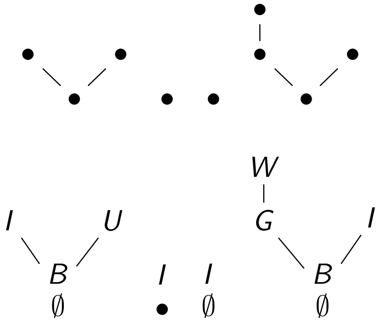

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