(Algebraic) Proof Theory for Substructural Logics and Applications

Agata Ciabattoni

Vienna University of Technology

agata@logic.at

Substructural logics

logics weaker than classical logic

- include
  - intuitionistic logic,
  - relevance logics,
  - linear logic without exponential,
  - fuzzy logics,
  - ...
- defined as extensions of Full Lambek calculus $\mathbf{FL}$
- useful for reasoning, e.g., about natural language, vagueness, resources, dynamic data structures, algebraic varieties ...

This talk

Structural rules

Rules admissibility

Display Logic

Sequent Calculi

Residuated Lattices

Dedekind Mac Neille Completions

Hypersequent Calculi

Hilbert axioms

Algebraic Equations

Residuation property

Cut -elimination

Standard Completeness

This talk

PART I : (towards a) systematic proof theory for substructural logics

PART II : an application of the introduced calculi

PART III : open problems and work in progress

Why proof theory?

Non-classical logics are often introduced using Hilbert calculi

- The applicability/usefulness of these logics strongly depends on the availability of analytic calculi.

(*praedicatum inest subjecto*)

Analytic calculi are

- useful for establishing various properties of logics
- key for developing automated reasoning methods.

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  Analytic calculi are

  - useful for establishing various properties of logics
  - key for developing automated reasoning methods.

- Favourite framework: Gentzen sequent calculus

Sequent Calculus

Sequents

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$$

Intuitively a sequent is understood as "the conjunction of $A_1, \ldots, A_n$ implies the disjunction of $B_1, \ldots, B_m$"

Sequent Calculus

Sequents

$$A_1, \ldots, A_n \Rightarrow B_1$$

Intuitively a sequent is understood as the multiset $\{A_1, \ldots, A_n\}$ implies B_1

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Axioms

E.g., $A \Rightarrow A$

Rules

- Logical (left and right)
- Structural
  E.g. (contraction, exchange and weakening)

$$\frac{\Gamma, A, A \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (c, l) \quad \frac{\Gamma, B, A \Rightarrow \Pi}{\Gamma, A, B \Rightarrow \Pi}\ (e, l) \quad \frac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (w, l)$$

- Cut

Sequent Calculus: the rule Cut

$$\boxed{\dfrac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ Cut}$$

(+) corresponds to transitivity in algebras: if $x \leq y$ and $y \leq z \Longrightarrow x \leq z$

(+) key to prove completeness w.r.t. Hilbert system

$$\text{modus ponens} \quad \frac{A \quad A \to B}{B}$$

(-) bad for proof search

Cut-elimination theorem

Each proof using Cut can be transformed into a proof without Cut.

The system FLe

- FLe $\approx$ commutative Lambek calculus
- FLe $\approx$ intuitionistic logic without weakening and contraction
- FLe $\approx$ intuitionistic Linear Logic without exponentials

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Algebraic semantics:
A *(bounded pointed) commutative residuated lattice* (FLe algebra) is

$$\mathbf{P} = \langle P, \wedge, \vee, \otimes, \to, \top, \mathbf{0}, \mathbf{1}, \bot \rangle$$

1. $\langle P, \wedge, \vee \rangle$ is a lattice with $\top$ greatest and $\bot$ least
2. $\langle P, \otimes, \mathbf{1} \rangle$ is a commutative monoid.
3. For any $x, y, z \in P$, $x \otimes y \leq z \iff y \leq x \to z$
4. $\mathbf{0} \in P$.

Notation: We write $a \leq b$ instead of $a = a \wedge b$ ($a \vee b = a$).

The system FLe

- FLe $\approx$ commutative Lambek calculus
- FLe $\approx$ intuitionistic logic without weakening and contraction
- FLe $\approx$ intuitionistic Linear Logic without exponentials

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FLe-algebras are *varieties*.

The system **FLe**

$$\frac{A, B, \Gamma \Rightarrow \Pi}{A \otimes B, \Gamma \Rightarrow \Pi}\ \otimes l \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}\ \otimes r$$

$$\frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow \Pi}{\Gamma, A \to B, \Delta \Rightarrow \Pi} \to l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B} \to r$$

$$\frac{A, \Gamma \Rightarrow \Pi \quad B, \Gamma \Rightarrow \Pi}{A \vee B, \Gamma \Rightarrow \Pi}\ \vee l \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ \vee r \qquad \frac{}{\mathbf{0} \Rightarrow}\ \mathbf{0}l$$

$$\frac{A_i, \Gamma \Rightarrow \Pi}{A_1 \wedge A_2, \Gamma \Rightarrow \Pi}\ \wedge l \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge r \qquad \frac{}{\Gamma \Rightarrow \top}\ \top r$$

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathbf{0}}\ \mathbf{0}r \qquad \frac{}{\Rightarrow \mathbf{1}}\ \mathbf{1}r \qquad \frac{}{\bot, \Gamma \Rightarrow \Pi}\ \bot l \qquad \frac{\Gamma \Rightarrow \Pi}{\mathbf{1}, \Gamma \Rightarrow \Pi}\ \mathbf{1}l$$

On the sequent calculus **FLe**

- For any set $\mathcal{A} \cup \{A\}$ of formulas,

$$\mathcal{A} \vdash_{\mathbf{FLe}} A \text{ iff } \varepsilon[\mathcal{A}] \models_{\mathsf{FLe}} \varepsilon(A)$$

  where $\varepsilon(-)$ is the equation corresponding to $-$.

- Theorem
  Any sequent provable in $\mathbf{FLe}$ is provable without using (Cut).

Commutative substructural logics

Are defined by adding equations to FLe-algebras or Hilbert axioms to the sequent calculus **FLe**.

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Example:
Gödel logic is obtained by adding

- the Hilbert axiom $(\alpha \to \beta) \vee (\beta \to \alpha)$ to intuitionistic logic (**FL** + exchange, weakening and contraction), or
- prelinearity $1 \leq (x \to y) \vee (y \to x)$ to Heyting algebras

Commutative substructural logics

Are defined by adding equations to FLe-algebras or Hilbert axioms to the sequent calculus **FLe**.

Cut-elimination is not preserved when axioms are added

A sequent calculus without cut-elimination is like a car without an engine (J-Y.Girard)

Axioms vs Rules

Example

- Contraction: $\alpha \to \alpha \otimes \alpha$ $\qquad \dfrac{A, A, \Gamma \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi}\ (c)$
- Weakening l: $\alpha \to 1$ $\qquad \dfrac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (w, l)$
- Weakening r: $0 \to \alpha$ $\qquad \dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ (w, r)$

They are *equivalent*, i.e.

$$\vdash_{\mathbf{FLe}+(axiom)} \quad = \quad \vdash_{\mathbf{FLe}+(rule)}$$

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$\boxed{\text{For which axioms can we do it?}}$

Order Theoretic Completions

- A completion of an algebra $\mathbf{A}$ is a complete algebra $\mathbf{B}$ (i.e. it has arbitrary $\bigvee$ and $\bigwedge$) such that $\mathbf{A} \subseteq \mathbf{B}$.
- Completions are not unique: filter/ideal extensions, canonical extensions, Dedekind-MacNeille completions, ...

Order Theoretic Completions

Dedekind Completion of Rationals

- For any $X \subseteq \mathbb{Q}$,

$$
\begin{aligned}
X^{\triangleright} &= \{y \in \mathbb{Q} : \forall x \in X . x \leq y\} \\
X^{\triangleleft} &= \{y \in \mathbb{Q} : \forall x \in X . y \leq x\}
\end{aligned}
$$

- X is closed if $X = X^{\triangleright\triangleleft}$
- $(\mathbb{Q}, +, \cdot)$ can be embedded into $(\mathcal{C}(\mathbb{Q}), +, \cdot)$ with

$$\mathcal{C}(\mathbb{Q}) = \{X \subseteq \mathbb{Q} : X \text{ is closed}\}$$

Dedekind completion extends to various ordered algebras (MacNeille).

Closure under DM completion

Although the DM completion applies to all individual FLe algebras, it may produce an FLe algebra that is not in a given variety, containing the original one.

Hence a natural question is:

- Given a variety of FLe-algebras, is it closed under DM completion?

or equivalently

- Given an equation over commutative residuated lattices, is it preserved by DM completion?

The two questions

- Given an equation over commutative residuated lattices, is it preserved by DM completion?
- Given an Hilbert axiom over FLe, can it be transformed into a rule that preserve cut-elimination?

Are they related?

... algebraic proof theory (AC, Galatos and Terui)

From axioms/equations to rules: the ingredients

- Use of the invertible rules of the base calculus ($\mathbf{FLe}$)
- Use of the Ackermann Lemma
  An algebraic equation $t \leq u$ is equivalent to a quasiequation $u \leq x \Longrightarrow t \leq x$, and also to $x \leq t \Longrightarrow x \leq u$, where x is a fresh variable not occurring in t, u.

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Example: $\Rightarrow (x \to y) \to z$

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$$x \to y \Rightarrow z$$

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Example: $\Rightarrow (x \rightarrow y) \rightarrow z$ is equivalent to ($(\rightarrow, r)$ is invertible)

$$x \rightarrow y \Rightarrow z$$

By Ackermann Lemma (A new metavariable)

$$\frac{A \Rightarrow x \rightarrow y}{A \Rightarrow z}$$

Classification

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The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas (equations) defined by:

$$\mathcal{P}_0,\ \mathcal{N}_0 := \text{Atomic formulas}$$

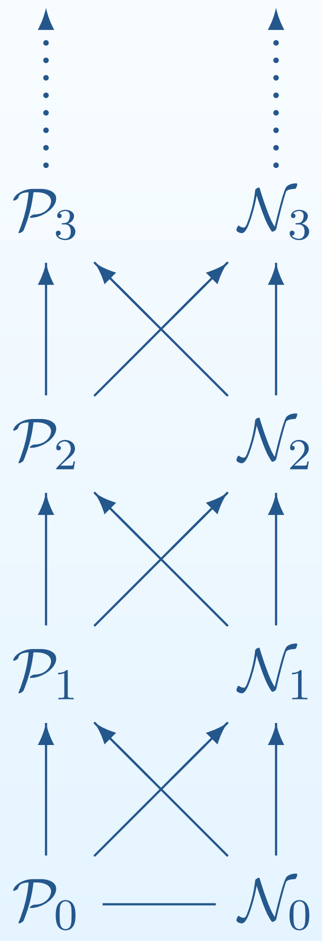
$$\mathcal{P}_{n+1} := \mathcal{N}_n \mid \mathcal{P}_{n+1} \otimes \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid 1 \mid \perp$$

$$\mathcal{N}_{n+1} := \mathcal{P}_n \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid 0 \mid \top$$

$\mathcal{P}$ and $\mathcal{N}$

- Positive connectives $\mathbf{1}, \perp, \otimes, \vee$ have *invertible left rules*:
- Negative connectives $\top, \mathbf{0}, \wedge, \to$ have *invertible right rules*:

Classification



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$\mathcal{P}$ and $\mathcal{N}$

- Positive connectives $\mathbf{1}, \bot, \otimes, \vee$ have *invertible left rules*:
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Examples of axioms/equations

Class	Axiom	Name
$\mathcal{N}_2$	$\alpha \to 1,\ 0 \to \alpha$	weakening
	$\alpha \to \alpha \otimes \alpha$	contraction
	$\alpha \otimes \alpha \to \alpha$	expansion
	$\otimes\alpha^n \to \otimes\alpha^m$	knotted axioms ($n, m \geq 0$)
	$\neg(\alpha \wedge \neg\alpha)$	weak contraction
$\mathcal{P}_2$	$\alpha \vee \neg\alpha$	excluded middle
	$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
$\mathcal{P}_3$	$\neg\alpha \vee \neg\neg\alpha$	weak excluded middle
	$\neg(\alpha \otimes \beta) \vee (\alpha \wedge \beta \to \alpha \otimes \beta)$	(wnm)
$\mathcal{N}_3$	$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$	Łukasiewicz axiom
	canonical formulas	Bezhanishvili, Galatos, Spada

Our preliminary results

$$
\begin{tikzcd}
{} & {} \\
\mathcal{P}_3 \arrow[u, dotted] & \mathcal{N}_3 \arrow[u, dotted] \\
\mathcal{P}_2 \arrow[u] \arrow[ur] & \color{red}{\mathcal{N}_2} \arrow[u] \arrow[ul] \\
\mathcal{P}_1 \arrow[u] \arrow[ur] & \mathcal{N}_1 \arrow[u] \arrow[ul] \\
\mathcal{P}_0 \arrow[u] \arrow[ur] \arrow[r, dash] & \mathcal{N}_0 \arrow[u] \arrow[ul]
\end{tikzcd}
$$

Algorithm to transform:

- axioms up to the class $\mathcal{N}_2$ into "good" structural rules in sequent calculus
- equations up to $\mathcal{N}_2$ into "good" quasiequations

$$t_1 \leq u_1 \text{ and..and } t_m \leq u_m \Longrightarrow t_{m+1} \leq u_{m+1}$$

(AC, N. Galatos and K. Terui). LICS 2008 and APAL 2012

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- axioms up to the class $\mathcal{N}_2$ into "good" structural rules in sequent calculus
- equations up to $\mathcal{N}_2$ into "good" quasiequations

Moreover

- analytic calculi iff DM completion
- in presence of weakening/integrality all axioms/equations up to $\mathcal{N}_2$ are tamed

(AC, N. Galatos and K. Terui). LICS 2008 and APAL 2012

Expressive power of structural sequent rules

Consider e.g.

$$(\alpha \to \beta) \vee (\beta \to \alpha) \in \mathcal{P}_2$$

Gödel logic := $\mathbf{IL}$ + $(\alpha \to \beta) \vee (\beta \to \alpha)$

Expressive power of structural sequent rules

Consider e.g.

$$(\alpha \to \beta) \lor (\beta \to \alpha) \in \mathcal{P}_2$$

- Can we find equivalent *good* structural sequent rules?

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Theorem
Each good (i.e. analytic) structural sequent rule is equivalent to an equation which is preserved by Dedekind MacNeille completions in presence of integrality.

(AC, N. Galatos and K. Terui. APAL 2012)

Expressive power of structural sequent rules

Consider e.g.

$$(\alpha \to \beta) \vee (\beta \to \alpha) \in \mathcal{P}_2$$

- Can we find equivalent *good* structural sequent rules?

Theorem (Proof Theory)
Any structural rule is either derivable in Gentzen's $\mathbf{LJ}$ or derives every formula in $\mathbf{LJ}$.

... it reminds

Theorem (Algebra) (Bezhanishvili & Harding 04)
$\mathcal{BA}$ is the only nontrivial proper subvariety of $\mathcal{HA}$ closed under DM completions.

Beyond sequent calculus

- Many useful and interesting equations have no equivalent structural sequent rules

Beyond sequent calculus

- Many useful and interesting equations have no equivalent structural sequent rules
- Many useful and interesting logics seem do not fit comfortably into the sequent framework.

A large range of variants and extensions have been indeed introduced. E.g.

Hypersequent Calculi,

Display calculi,

Labelled Deductive Systems,

Nested Calculi,

Bunched Calculi,

...

Hypersequent calculus

It is obtained embedding sequents into hypersequents

$$\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$$

where for all $i = 1, \ldots n$, $\Gamma_i \Rightarrow \Pi_i$ is a sequent.

Hypersequent calculus

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ Cut \qquad \frac{}{A \Rightarrow A}\ Identity$$

$$\frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow \Pi}{\Gamma, A \rightarrow B, \Delta \Rightarrow \Pi} \rightarrow l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \rightarrow r$$

Hypersequent calculus

$$\frac{G|\Gamma \Rightarrow A \quad G|A, \Delta \Rightarrow \Pi}{G|\Gamma, \Delta \Rightarrow \Pi}\ Cut \qquad \frac{}{G|A \Rightarrow A}\ Identity$$

$$\frac{G|\Gamma \Rightarrow A \quad G|B, \Delta \Rightarrow \Pi}{G|\Gamma, A \to B, \Delta \Rightarrow \Pi} \to l \qquad \frac{G|A, \Gamma \Rightarrow B}{G|\Gamma \Rightarrow A \to B} \to r$$

and adding suitable rules to manipulate the additional layer of structure.

$$\frac{G}{G \mid \Gamma \Rightarrow A}\ (\mathrm{ew}) \qquad \frac{G \mid \Gamma \Rightarrow A \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A}\ (\mathrm{ec})$$

Structural rules: an example

$$\frac{G \,|\, \Gamma, \Sigma' \Rightarrow \Delta' \quad G \,|\, \Gamma', \Sigma \Rightarrow \Delta}{G \,|\, \Gamma, \Sigma \Rightarrow \Delta \,|\, \Gamma', \Sigma' \Rightarrow \Delta'} \; (com)$$

(Avron, Annals of Math and art. Intell. 1991)

Gödel logic = $\mathbf{IL}$ + $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$

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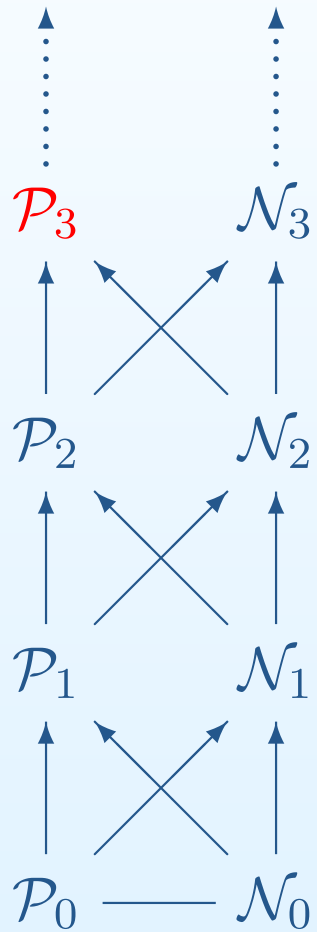
Gödel logic = $\mathbf{IL}$ + $(\alpha \to \beta) \vee (\beta \to \alpha)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\beta \Rightarrow \beta \qquad \alpha \Rightarrow \alpha}{\alpha \Rightarrow \beta \mid \beta \Rightarrow \alpha}\,(\mathrm{com})}{\alpha \Rightarrow \beta \mid\ \Rightarrow \beta \to \alpha}\,(\to,\mathrm{r})}{\Rightarrow \alpha \to \beta \mid\ \Rightarrow \beta \to \alpha}\,(\to,\mathrm{r})}{\Rightarrow \alpha \to \beta \mid\ \Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha)}\,(\vee_{\mathrm{i}},\mathrm{r})}{\Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha) \mid\ \Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha)}\,(\vee_{\mathrm{i}},\mathrm{r})}{\Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha)}\,(\mathrm{EC})$$

Climbing up the hierarchy

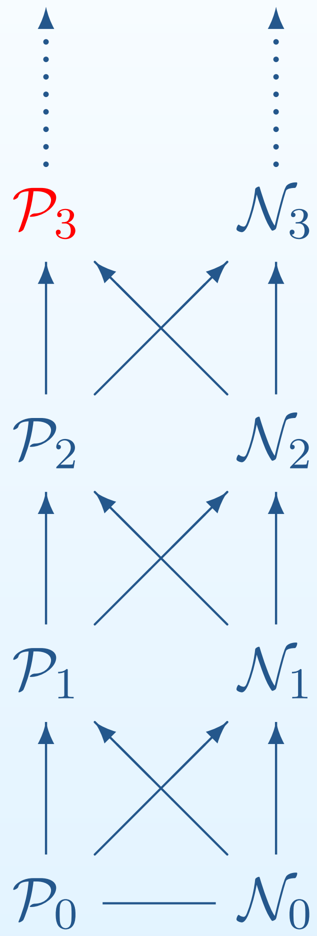


Algorithm to transform:

- axioms up to the class $\mathcal{P}_3'$ into "good" structural rules in hypersequent calculus
- equations up to $\mathcal{P}_3'$ into "good" analytic clauses

$$t_1 \leq u_1 \text{ and..and } t_m \leq u_m \Rightarrow t_{m+1} \leq u_{m+1} \text{ or..or } t_n \leq u_n$$

Climbing up the hierarchy



Algorithm to transform:

- axioms up to the class $\mathcal{P}'_3$ into "good" structural rules in hypersequent calculus
- equations up to $\mathcal{P}'_3$ into "good" analytic clauses

Moreover

- equations up to $\mathcal{P}'_3$ preserved by DM completions when applied to s.i. algebras
- analytic calculi iff HyperDM completion
- axioms/equations up to $\mathcal{P}_3$ are tamed in presence of integrality

(AC, N. Galatos and K. Terui). Algebra Universalis, 2011, and Submitted 2014.

From axioms to rules: an example

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and to

$$\frac{}{G \mid \alpha \Rightarrow \beta \mid \beta \Rightarrow \alpha}$$

From axioms to rules: an example

$$(\alpha \to \beta) \vee (\beta \to \alpha)$$

$$\frac{}{G \,|\, \alpha \Rightarrow \beta \,|\, \beta \Rightarrow \alpha}$$

by Ackermann Lemma: Any sequent $\alpha' \Rightarrow \beta'$ is equivalent to

$$\frac{\Gamma \Rightarrow \alpha'}{\Gamma \Rightarrow \beta'} \quad \text{and also to} \quad \frac{\beta', \Gamma \Rightarrow \Delta}{\alpha', \Gamma \Rightarrow \Delta}$$

(for Γ, Δ fresh meta-variables)

From axioms to rules: an example

$$(\alpha \to \beta) \vee (\beta \to \alpha)$$

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(for Γ, Δ fresh meta-variables) is equivalent to

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is equivalent to

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From axioms to rules: an example

$$(\alpha \to \beta) \vee (\beta \to \alpha)$$

$$\frac{}{G \mid \alpha \Rightarrow \beta \mid \beta \Rightarrow \alpha}$$

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is equivalent to

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(Avron, Annals of Math and Art. Intell. 1991)

To sum up

- systematic generation of good (hyper)sequent rules equivalent to axioms up to $\mathcal{P}'_3$ ($\mathcal{P}_3$ in presence of weakening)
- identification/introduction of appropriate completions that work for equations up to the level $\mathcal{P}'_3$ ($\mathcal{P}_3$ in presence of weakening)

http://www.logic.at/staff/lara/tinc/webaxiomcalc/

This talk

PART I : (towards a) systematic proof theory for substructural logics

PART II : an application of the introduced calculi

PART III : open problems and work in progress

Standard Completeness

Completeness of axiomatic systems with respect to algebras whose lattice reduct is the real unit interval $[0,1]$.

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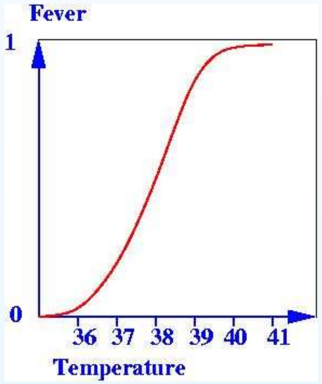
(Hajek 1998) Formalizations of *Fuzzy Logic*



Uninorms and t-norms

- A uninorm is a function $* : [0,1]^2 \to [0,1]$ satisfying, $\forall x, y, z \in [0,1]$:
  - $x * y = y * x$ (Commutativity),
  - $(x * y) * z = x * (y * z)$ (Associativity),
  - $x \leq y$ implies $x * z \leq y * z$ (Monotonicity),
  - $e \in [0,1] \quad e * x = x$ (Identity).

  The *residuum* is a function $\Rightarrow_*: [0,1]^2 \to [0,1]$ where $x \Rightarrow_* y = max\{z \mid x * z \leq y\}$.
- A t-norm is a uninorm in which $e = 1$.

Uninorm and t-norm based logics

We fix a propositional language with connectives $\wedge, \vee, \otimes, \to$ and constants $\bot, \top, e, f$.
Evaluation $v\colon Var \to [0,1]$ extend inductively over all formulas:

$$v(A \to B) = v(A) \Rightarrow_* v(B) \quad v(A \wedge B) = v(A) * v(B)$$

$$v(\bot) = 0 \quad v(\top) = 1 \quad v(f), v(e) \in [0,1]$$

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Evaluation $v \colon Var \rightarrow \; [0,1]$

- Gödel logic

  $v(A \wedge B) = \min\{v(A), v(B)\}$

  $v(A \vee B) = \max\{v(A), v(B)\}$

  $v(A \rightarrow B) = 1$ if $v(A) \leq v(B)$, and $v(B)$ otherwise

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- UL Uninorm logic (Metcalfe, Montagna 2007)

  $v(A \otimes B) = v(A) * v(B)$, $\quad *$ left continuous uninorm

  $v(A \rightarrow B) = v(A) \rightarrow_* v(B)$ $\quad \rightarrow_*$ residuum of $*$

- MTL Monoidal T-norm logic (Godo, Esteva 2001)

  $v(A \otimes B) = v(A) * v(B)$, $\quad *$ left continuous t-norm

  $v(A \rightarrow B) = v(A) \rightarrow_* v(B)$ $\quad \rightarrow_*$ residuum of $*$

Uninorm or t-norm -based logics II

Often described by *adding* or *removing* axioms to already known logics.

Example

- UL = $\underline{\text{FLe with } ((\alpha \to \beta) \wedge e) \vee ((\beta \to \alpha) \wedge e)}$ (prelinearity)
- MTL = $\underline{\text{UL with } \alpha \to e \text{ and } f \to \alpha}$ (weakening/integrality)
- Gödel logic = MTL with contraction $\alpha \to \alpha \otimes \alpha$
- UML = UL with contraction $\alpha \to \alpha \otimes \alpha$ and mingle $\alpha \otimes \alpha \to \alpha$
- WNM = MTL with $\neg(\alpha \otimes \beta) \vee (\alpha \wedge \beta \to \alpha \otimes \beta)$
- BL = MTL with divisibility $(\alpha \wedge \beta) \to (\alpha \otimes (\alpha \to \beta))$
- $\ldots$

Standard Completeness?

$\boxed{\text{Question}}$ Given a logic $\mathcal{L}$ (expressed Hilbert style) obtained by extending MTL or UL with

- $A \vee \neg A$ (excluded middle)?
- $A^{n-1} \to A^n$ (n-contraction)?
- $\neg(A \otimes B) \vee (A \wedge B \to A \otimes B)$ (weak nilpotent minimum)?
- ....

Is $\mathcal{L}$ standard complete? (*is it a formalization of Fuzzy Logic?*)

$\boxed{\text{usually case-by-case answer}}$

Algebraic Semantics

- A UL-algebra is an FLe algebra

$$\mathbf{P} = \langle P, \wedge, \vee, \otimes, \rightarrow, 1 \rangle$$

  satisfying

$$1 \leq ((x \rightarrow y) \wedge 1) \vee ((y \rightarrow x) \wedge 1) \quad \text{for all } x, y \in P$$

- An MTL-algebra is an integral UL-algebra ($x \leq 1$, for all $x \in P$)

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- An MTL-algebra is an integral UL-algebra ($x \leq 1$, for all $x \in P$)

Useful properties:

- UL and MTL- algebras are complete w.r.t. chains
- Lemma: For every chain $\mathbf{A}$ in FLe

$$\models_{\mathbf{A}} 1 \leq (t \wedge 1) \vee (u \wedge 1) \quad \text{iff} \quad \models_{\mathbf{A}} 1 \leq t \; or \; 1 \leq u$$

Standard Completeness – algebraic approach

Given a logic $\mathcal{L}$ extending UL with axiom α:

1. Identify the algebraic semantics of $\mathcal{L}$ ($\mathcal{L}$-algebras)
2. Show completeness of $\mathcal{L}$ w.r.t. linear, countable $\mathcal{L}$-algebras
3. Find an embedding of linear countable $\mathcal{L}$-algebras into linear dense countable $\mathcal{L}$-algebras
4. Dedekind-MacNeille style completion (embedding into $\mathcal{L}$-algebras with lattice reduct $[0, 1]$)

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   $UL + \alpha \quad \Longleftrightarrow \quad UL$-chains satisfying $1 \leq \alpha$
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3. Find an embedding of linear countable $\mathcal{L}$-algebras into linear dense countable $\mathcal{L}$-algebras
   i.e. whenever $x \not\leq y$, there exists z such that $x \not\leq z$ and $z \not\leq y$
4. Dedekind-MacNeille style completion (embedding into $\mathcal{L}$-algebras with lattice reduct $[0, 1]$)

Standard Completeness – algebraic approach

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4. Dedekind-MacNeille style completion (embedding into $\mathcal{L}$-algebras with lattice reduct $[0,1]$)
   - DM completions of a dense UL-chain is still a dense UL-chain (= it is preserved by DM-completions).
   - Prove that additional equations are preserved

Standard Completeness – algebraic approach

Given a logic $\mathcal{L}$ extending UL with axiom α:

1. Identify the algebraic semantics of $\mathcal{L}$ ($\mathcal{L}$-algebras)
2. Show completeness of $\mathcal{L}$ w.r.t. linear, countable $\mathcal{L}$-algebras
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- Step 3 (rational completeness): problematic (mainly ad hoc solutions)

Standard Completeness via proof theory

(Metcalfe, Montagna JSL 2007) $\mathcal{L}$ + (*density*) is rational complete:

$$\frac{(\Phi \to p) \vee (p \to \Psi) \vee \Xi}{(\Phi \to \Psi) \vee \Xi}\ (density)$$

where $p \notin \Phi, \Psi, \Xi$

Consider $\mathcal{L}$ + (*density*)

(Step 1) Show that density produces no new theorems (Rational completeness)

(Step 2) Dedekind-MacNeille style completion

Density vs Cut in hypersequent calculi

$$\boxed{\frac{(\Phi \to p) \vee (p \to \Psi) \vee \Xi}{(\Phi \to \Psi) \vee \Xi}\ (density)}$$

- $$\frac{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ (density)$$

  where p is does not occur in the conclusion.

- $$\frac{G \mid \Gamma \Rightarrow A \quad G \mid A \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ (cut)$$

Density elimination

- Similar to cut-elimination
- Proof by induction on the length of derivations

Density elimination

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- Proof by induction on the length of derivations

(-, Metcalfe TCS 2008) Given a density-free derivation, ending in

$$\dfrac{\begin{array}{c}\vdots\, d' \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\ (\mathrm{density})$$

Density elimination

(-, Metcalfe TCS 2008) Given a density-free derivation, ending in

$$\dfrac{\begin{matrix} \vdots\, d' \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta \end{matrix}}{G \mid \Gamma \Rightarrow \Delta}\ (\text{density})$$

$$\dfrac{\begin{matrix} \vdots\, d' \\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \end{matrix}}{G \mid \Gamma \Rightarrow \Delta}\ (\text{EC})$$

- Asymmetric substitution: p is replaced
  - With Δ when occuring on the right
  - With Γ when occuring on the left

Density elimination: problem with (com)

$$\dfrac{p \Rightarrow p \qquad \begin{array}{c}\vdots\\ \Pi \Rightarrow \Psi\end{array}}{\Pi \Rightarrow p \mid p \Rightarrow \Psi}\,(com) \qquad\qquad \dfrac{\Gamma \Rightarrow \Delta \qquad \begin{array}{c}\vdots\\ \Pi \Rightarrow \Psi\end{array}}{\Pi \Rightarrow \Delta \mid \Gamma \Rightarrow \Psi}\,(com)$$

$$\vdots\, d \qquad\qquad\qquad\qquad \vdots\, d^*$$

$$\dfrac{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\,(\mathrm{D}) \qquad\qquad \dfrac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\,(\mathrm{EC})$$

- $p \Rightarrow p$ axiom
- $\Gamma \Rightarrow \Delta$ not an axiom

Density elimination: problem with (com)

$$
\dfrac{\dfrac{p \Rightarrow p \qquad \begin{array}{c}\vdots\\ \Pi \Rightarrow \Psi\end{array}}{\Pi \Rightarrow p \mid p \Rightarrow \Psi}\,(com)}{\begin{array}{c}\vdots\, d\\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{array}} \qquad\qquad \dfrac{\cancel{\Gamma \Rightarrow \Delta} \qquad \begin{array}{c}\vdots\\ \Pi \Rightarrow \Psi\end{array}}{\Pi \Rightarrow \Delta \mid \Gamma \Rightarrow \Psi}\,(com)
$$

$$
\dfrac{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\,(\mathrm{D}) \qquad\qquad \dfrac{\begin{array}{c}\Pi \Rightarrow \Delta \mid \Gamma \Rightarrow \Psi\\ \vdots\, d^*\\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\,(\mathrm{EC})
$$

- $p \Rightarrow p$ axiom
- $\Gamma \Rightarrow \Delta$ not an axiom

Solution (with weakening)

(AC, Metcalfe 2008)

$$
\begin{array}{c}
\dfrac{p \Rightarrow p \qquad \begin{array}{c}\vdots\\ \Pi \Rightarrow \Psi\end{array}}{\Pi \Rightarrow p \mid p \Rightarrow \Psi}\;(com)\\
\vdots\; d\\
\dfrac{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\;\text{(D)}
\end{array}
\qquad\qquad
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\\ G \mid \Gamma \Rightarrow \overset{\Pi}{\cancel{p}} \mid \overset{\Pi}{\cancel{p}} \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\vdots\\ \Pi \Rightarrow \Psi\end{array}}{\Pi \Rightarrow \Delta \mid \Gamma \Rightarrow \Psi}\;(cut)\\
\vdots\; d^*\\
\dfrac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\;\text{(EC)}
\end{array}
$$

Step 2: general conditions for density elimination

- In presence of weakening/integrality
  Theorem (AC, Baldi TCS to appear)
  The hypersequent calculus for MTL + a large class of rules equivalent to axioms within the class $\mathcal{P}_3$ (**convergent** rules) admits density elimination

i.e. rules equivalent to axioms within the class $\mathcal{P}_3$ and whose premises do not mix "too much" the conclusion

Example :

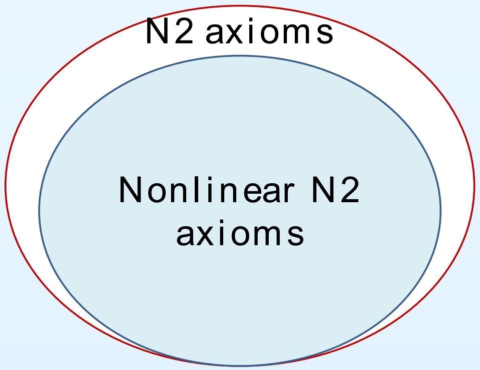
$$\frac{\begin{array}{cc} G \mid \Gamma_2, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \mid \Gamma_1, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \\ G \mid \Gamma_1, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \mid \Gamma_2, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \end{array}}{G \mid \Gamma_2, \Gamma_3 \Rightarrow \; \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1} \; (wnm)$$

Axiom: $\neg(\alpha \otimes \beta) \vee (\alpha \wedge \beta \to \alpha \otimes \beta)$

Step 2: general conditions for density elimination

- In presence of weakening/integrality
  Theorem (AC, Baldi TCS to appear)
  The hypersequent calculus for MTL + a large class of rules equivalent to axioms within the class $\mathcal{P}_3$ (**convergent** rules) admits density elimination

- Without weakening/integrality
  Theorem (AC, Baldi ISMVL 2015)
  The hypersequent calculus for UL + nonlinear rules (and/or mingle) admits density elimination



Recall: Standard Completeness via proof theory

(Metcalfe, Montagna JSL 2007) Given a logic $\mathcal{L}$:

(Step 1) $\boxed{\text{Show that density produces no new theorems}}$

(Step 2) Dedekind-MacNeille style completion

- DM completions of a dense UL-chain is still a dense UL-chain (= it is preserved by DM-completions).
- This holds for all $\mathcal{P}_3'$ equations ($\mathcal{P}_3$ in presence of weakening)

Example

Known Logics

- MTL + $\neg(\alpha \cdot \beta) \vee ((\alpha \wedge \beta) \to (\alpha \cdot \beta))$
- MTL + $\neg\alpha \vee \neg\neg\alpha$
- MTL + $\alpha^{n-1} \to \alpha^n$
- UL + $\alpha^{n-1} \to \alpha^n$
- ...

New Fuzzy Logics

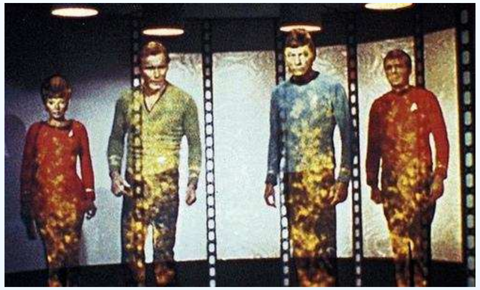
- MTL + $\neg(\alpha \cdot \beta)^n \vee ((\alpha \wedge \beta)^{n-1} \to (\alpha \cdot \beta)^n)$, for all $n > 1$
- UL + $\neg\alpha \vee \neg\neg\alpha$
- UL + $\alpha^m \to \alpha^n$
- ...

This talk

PART I : (towards a) systematic proof theory for substructural logics

PART II : an application of the introduced calculi

PART III : open problems



and intermediary results

Open problems I

$$
\begin{tikzcd}
{} & {} \\
\mathcal{P}_3 \arrow[u, dotted] & \mathcal{N}_3 \arrow[u, dotted] \\
\mathcal{P}_2 \arrow[u] \arrow[ur] & \mathcal{N}_2 \arrow[u] \arrow[ul] \\
\mathcal{P}_1 \arrow[u] \arrow[ur] & \mathcal{N}_1 \arrow[u] \arrow[ul] \\
\mathcal{P}_0 \arrow[u] \arrow[ur] \arrow[r, dash] & \mathcal{N}_0 \arrow[u] \arrow[ul]
\end{tikzcd}
$$

Uniform treatment of axioms in $\mathcal{N}_3$ and behond

Remark on $\mathcal{N}_3$: it contains

(a) equations that are not preserved under completions.

(b) all (axiomatizable) intermediate logics (via canonical formulas).

Negative results on (hyper)sequent rules

$$
\begin{tikzcd}
{} & {} \\
\mathcal{P}_3 \arrow[u, dotted] & \mathcal{N}_3 \arrow[u, dotted] \\
\mathcal{P}_2 \arrow[u] \arrow[ur] & \mathcal{N}_2 \arrow[u] \arrow[ul] \\
\mathcal{P}_1 \arrow[u] \arrow[ur] & \mathcal{N}_1 \arrow[u] \arrow[ul] \\
\mathcal{P}_0 \arrow[u] \arrow[ur] \arrow[r, dash] & \mathcal{N}_0 \arrow[u] \arrow[ul]
\end{tikzcd}
$$

Sequent structural rules: only equations

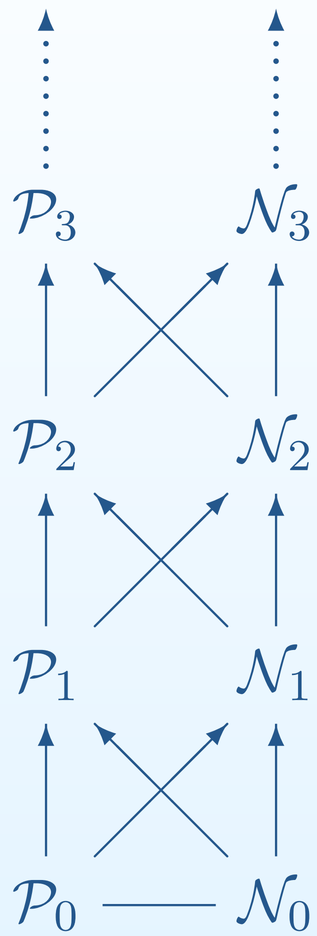
- closed under DM completion, with integrality
- that hold in Heyting algebras (IL)

Hypersequent structural rules: only equations

- closed under HyperDM completions, with integrality
- that hold in Heyting algebras generated by the 3-element algebras or derive $1 \leq x \vee \neg x^n$ in FLew

(AC, N. Galatos and K. Terui. Submitted 2014)

Open problems I

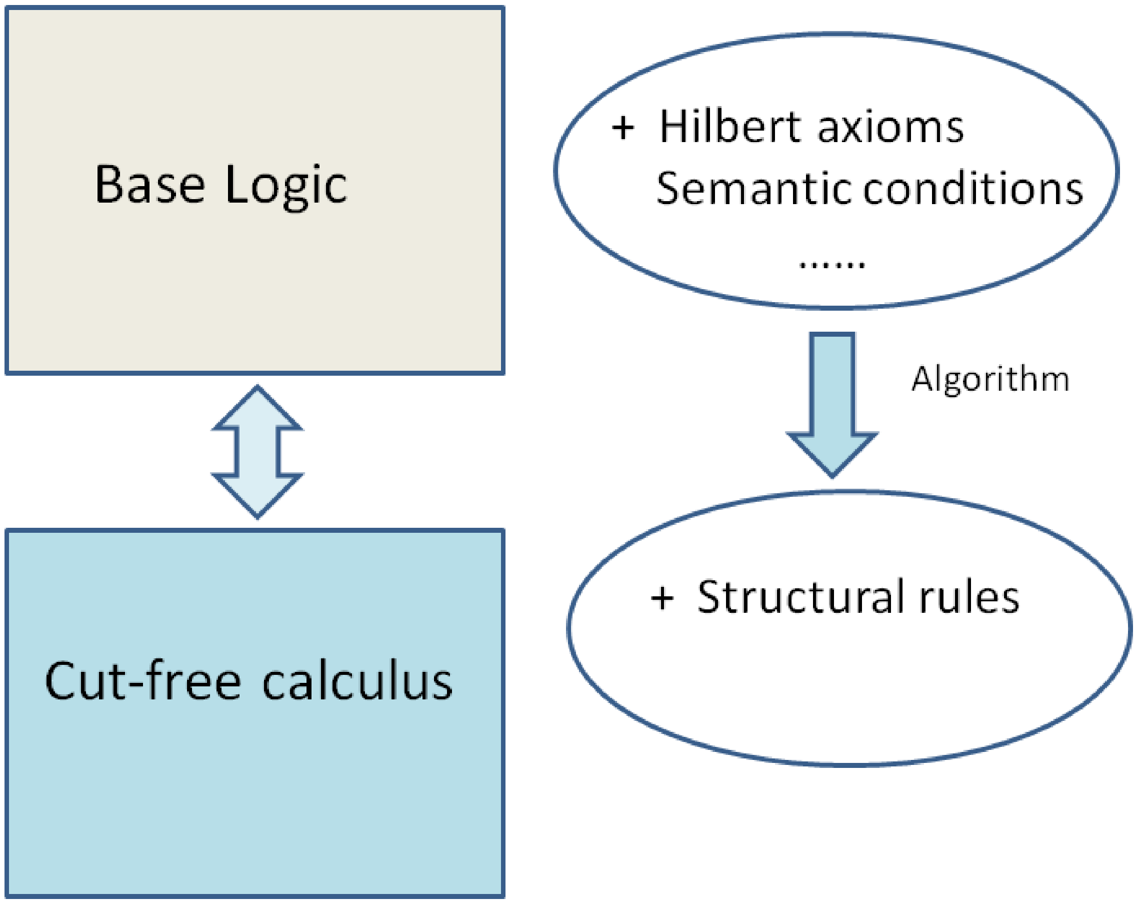


Uniform treatment of axioms in $\mathcal{N}_3$ and behond

Partial answers:

- generation of logical rules
  - **Ł**ukasiewicz equation ($\subseteq \mathcal{N}_3$)
  - Bd_2 axiom ($\subseteq \mathcal{P}_4$)
  - ....
- adopting formalisms more complex than the (hyper)sequent calculus

From axioms to rules: general idea



Display Calculus

Gentzen Sequent: $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$

$(A_1 \wedge \ldots \wedge A_n \Rightarrow B_1 \vee \ldots \vee B_m)$

Display Calculus

Gentzen Sequent: $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$

$$(A_1 \wedge \ldots \wedge A_n \Rightarrow B_1 \vee \ldots \vee B_m)$$

Belnap's idea ('82) : look at $\Rightarrow$ as a deducibility relation between finite possible complex data (structures)

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Display Sequent

$X \Rightarrow Y$, where X, Y are structures which are built from formulae using structural connectives.

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Belnap's idea ('82) : look at $\Rightarrow$ as a deducibility relation between finite possible complex data (structures)

Display Sequent

$X \Rightarrow Y$, where X, Y are structures which are built from formulae using structural connectives.

> Display property: given a display sequent $X \Rightarrow Y$ and any occurrence of a substructure Z in the sequent, that occurrence can be displayed as
>
> $$Z \Rightarrow U \text{ or as } U \Rightarrow Z$$
>
> using structural rules (*display rules*).

Cut elimination in display calculus

Enjoyed by all calculi satisfying some syntactic conditions (C1 – C8)

- Subformula property (C1)
- For structural rules: each structure variable letter (i) is unique in conclusion (ii) has same polarity in each occurrence and (iii) is closed under arbitrary substitution.
- For logical rules: a cut in which the cut formula is principal in both premises, can be replaced by smaller cuts (C8)

Only C8 is non-trivial to verify (C8 not applicable to structural rules)

Axioms vs rules in display calculus

- Given any *suitable* display calculus C for a logic $\mathcal{L}$
- Classify the formulas of $\mathcal{L}$ according to the invertible rules of C
- Apply the algorithm using the same ingredients

- the invertible rules of C
- Ackermann's lemma

$$\boxed{\frac{\mathcal{S}}{X \vdash A}\rho_1 \qquad \frac{\mathcal{S} \quad A \vdash M}{X \vdash M}\rho_2} \qquad \boxed{\frac{\mathcal{S}}{A \vdash X}\delta_1 \qquad \frac{\mathcal{S} \quad M \vdash A}{M \vdash X}\delta_2}$$

Axioms vs rules in display calculus

- Given any *suitable* display calculus C for a logic $\mathcal{L}$
- Classify the formulas of $\mathcal{L}$ according to the invertible rules of C
- Apply the algorithm using the same ingredients

Our results

Theorem (AC, Ramanayake 2013)
Let C be an **amenable** calculus for $\mathcal{L}$ and A be an **I2 acyclic** axiom. There is an **analytic rule extension** for $\mathcal{L} + A$.

Theorem (AC, Ramanayake Submitted 2014)
Let C be an **amenable** and **well-behaved** calculus for $\mathcal{L}$ and let $\mathcal{L}'$ be an axiomatic extension of $\mathcal{L}$. Then there is an analytic rule extension of C for $\mathcal{L}'$ iff $\mathcal{L}'$ is an extension of $\mathcal{L}$ by **I2 acyclic** axioms.

Suitable display calculi?

Amenable calculi satisfy the following conditions:

1. The structural connectives are interpretable as connective of the logic
2. There are binary connectives $\vee$ and $\wedge$ that are commutative and associative and Moreover
   (a)$_\vee$ $A \vdash Y$ and $B \vdash Y$ implies $A \vee B \vdash Y$
   (b)$_\vee$ $X \vdash A$ implies $X \vdash A \vee B$ for any formula B.
   (a)$_\wedge$ $X \vdash A$ and $X \vdash B$ implies $X \vdash A \wedge B$
   (b)$_\wedge$ $A \vdash Y$ implies $A \wedge B \vdash Y$ for any formula B.
3. There are logical constants c_a, c_b such that the following sequents are derivable for arbitrary structures X and Y:

$$c_a \vdash Y \qquad\qquad X \vdash c_s$$

Algorithm for display calculi

The procedure to transform axioms into structural rules for display calculi is language and logic independent. It works for

- all display calculi satisfying purely syntactic conditions
- for a large class of axioms depending only on the invertible rules of the base calculus

An instantiation: back to substructural logics .....

The logic BiFLe

The logic BiFLe

Algebraic Semantics

A structure $\mathcal{A} = (A, \vee, \wedge, \otimes, \rightarrow, 1, \oplus, \prec\!\!\!\!-\, , 0)$ is a *commutative Bi-Lambek algebra* BiFL_e if:

1. $(A, \vee, \wedge)$ is a lattice
2. (a) $(A, \otimes, 1)$ is a commutative monoid
   (b) $(A, \oplus, 0)$ is a commutative monoid
3. Residuation properties:
   (a) $x \otimes y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightarrow z$, for every $x, y, z \in A$
   (b) $z \leq x \oplus y$ iff $z \prec\!\!\!\!- \, x \leq y$ iff $z \prec\!\!\!\!- \, y \leq x$, for every $x, y, z \in A$.

Residuation property and display rules

From the residuation properties (for every $x, y, z \in A$)

$$\overbrace{x \otimes y}^{,} \leq z \quad \Leftrightarrow \quad x \leq \overbrace{y \to z}^{>} \quad \Leftrightarrow \quad y \leq \overbrace{x \to z}^{>}$$

$$z \leq \underbrace{x \oplus y}_{;} \quad \Leftrightarrow \quad \underbrace{z \prec x}_{<} \leq y \quad \Leftrightarrow \quad \underbrace{z \prec y}_{<} \leq x$$

to residuation rules (display rules)

$$\frac{\dfrac{X, Y \vdash Z}{X \vdash Y > Z}}{Y \vdash X > Z} \qquad \frac{\dfrac{Z \vdash X ; Y}{Z < X \vdash Y}}{Z < Y \vdash X}$$

Residuation property and display rules

to residuation rules (display rules)

$$\frac{\dfrac{X, Y \vdash Z}{X \vdash Y > Z}}{Y \vdash X > Z} \qquad \frac{\dfrac{Z \vdash X ; Y}{Z < X \vdash Y}}{Z < Y \vdash X}$$

- Example: *display* the occurrence of r in $(p < q; r), s \vdash z$:

$$\cfrac{\cfrac{\cfrac{\cfrac{(p < q) < (s > z) \vdash r}{p < q \vdash r; (s > z)}}{p \vdash q; r; (s > z)}}{p < (q; r) \vdash s > z}}{(p < q; r), s \vdash z}$$

Display calculus rules

- Rewrite rules:

$$\frac{A, B \vdash Y}{A \otimes B \vdash Y} \otimes\mathsf{l} \qquad \frac{X \vdash A ; B}{X \vdash A \oplus B} \oplus\mathsf{r} \qquad \begin{array}{c} L, \Phi \vdash M \\ \hline\hline L \vdash M \\ \hline\hline \Phi, L \vdash M \end{array}$$

$$\frac{A < B \vdash Y}{A \prec B \vdash Y} \prec\mathsf{l} \qquad \frac{X \vdash A > B}{X \vdash A \rightarrow B} \rightarrow\mathsf{r} \qquad \begin{array}{c} L \vdash M ; \Phi \\ \hline\hline L \vdash M \\ \hline\hline L \vdash \Phi ; M \end{array}$$

Display calculus rules

- Rewrite rules:

$$\frac{A, B \vdash Y}{A \otimes B \vdash Y} \otimes\mathsf{l} \qquad \frac{X \vdash A; B}{X \vdash A \oplus B} \oplus\mathsf{r} \qquad \dfrac{\dfrac{L, \Phi \vdash M}{L \vdash M}}{\Phi, L \vdash M}$$

$$\frac{A < B \vdash Y}{A \prec B \vdash Y} \prec\mathsf{l} \qquad \frac{X \vdash A > B}{X \vdash A \to B} \to\mathsf{r} \qquad \dfrac{\dfrac{L \vdash M; \Phi}{L \vdash M}}{L \vdash \Phi; M}$$

- Structural rules:

$$\frac{X, Y \vdash Z}{Y, X \vdash Z}\,\mathsf{le} \qquad \frac{X \vdash Y, Z}{X \vdash Z, Y}\,\mathsf{re}$$

$$\frac{X, (Y, Z) \vdash U}{(X, Y), Z \vdash U}\,\mathsf{la} \qquad \frac{X \vdash (U; V); W}{X \vdash U; (V; W)}\,\mathsf{ra}$$

Display calculus rules

- Rewrite rules:

$$\frac{A, B \vdash Y}{A \otimes B \vdash Y} \otimes\mathsf{l} \qquad \frac{X \vdash A; B}{X \vdash A \oplus B} \oplus\mathsf{r} \qquad \cfrac{\cfrac{L, \Phi \vdash M}{L \vdash M}}{\Phi, L \vdash M}$$

$$\frac{A < B \vdash Y}{A \prec B \vdash Y} \prec\mathsf{l} \qquad \frac{X \vdash A > B}{X \vdash A \to B} \to\mathsf{r} \qquad \cfrac{\cfrac{L \vdash M; \Phi}{L \vdash M}}{L \vdash \Phi; M}$$

- Obtain missing decoding rules:

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \otimes B} \otimes\mathsf{r} \qquad \frac{A \vdash X \qquad B \vdash Y}{A \oplus B \vdash X; Y} \oplus\mathsf{l}$$

$$\frac{X \vdash A \qquad B \vdash Y}{X < Y \vdash A \prec B} \prec\mathsf{r} \qquad \frac{X \vdash A \qquad B \vdash Y}{A \to B \vdash X > Y} \to\mathsf{l}$$

From axioms to rules for substructural logics

.. Let us consider intermediate logics

- Sequent Calculus: class $\mathcal{N}_2$
- Hypersequent Calculus: class $\mathcal{P}_3$

w.r.t.

$$\mathcal{P}_{n+1} := \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid \bot$$

$$\mathcal{N}_{n+1} := \mathcal{P}_n \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid \top$$

With display calculus we capture $\mathcal{N}_2^d \quad (\supset \mathcal{P}_3)$, where

$$\mathcal{P}_{n+1}^d := \mathcal{N}_n^d \mid \mathcal{P}_{n+1}^d \wedge \mathcal{P}_{n+1}^d \mid \mathcal{P}_{n+1}^d \vee \mathcal{P}_{n+1}^d \mid \bot$$

$$\mathcal{N}_{n+1}^d := \mathcal{P}_n^d \mid \mathcal{P}_{n+1}^d \rightarrow \mathcal{N}_{n+1}^d \mid \mathcal{N}_{n+1}^d \wedge \mathcal{N}_{n+1}^d \mid \mathcal{N}_{n+1}^d \vee \mathcal{N}_{n+1}^d \mid \top$$

Our algorithm: an example

$$\vdash (\alpha \to \beta) \vee (\beta \to \alpha)$$

Our algorithm: an example

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$$\frac{X \vdash A, B}{X \vdash A \vee B}\ (\vee r) \qquad I \vdash \alpha \to \beta, \beta \to \alpha$$

Our algorithm: an example

$$\vdash (\alpha \to \beta) \vee (\beta \to \alpha)$$

$$I \vdash \alpha \to \beta, \beta \to \alpha$$

by display rules $\quad I < (\alpha \to \beta) \vdash \beta \to \alpha$

Our algorithm: an example

$$\vdash (\alpha \to \beta) \vee (\beta \to \alpha)$$

$$I \vdash \alpha \to \beta, \beta \to \alpha$$

by display rules $\quad I < (\alpha \to \beta) \vdash \beta \to \alpha$

$$\frac{X \vdash A > B}{X \vdash A \to B}\ (\to r) \qquad\qquad I < (\alpha \to \beta) \vdash \beta > \alpha$$

Our algorithm: an example

$$\vdash (\alpha \to \beta) \vee (\beta \to \alpha)$$

$$I \vdash \alpha \to \beta, \beta \to \alpha$$

by display rules $\quad I < (\alpha \to \beta) \vdash \beta \to \alpha$

$$I < (\alpha \to \beta) \vdash \beta > \alpha$$

and $\quad I < (\beta > \alpha) \vdash \alpha > \beta$

Our algorithm: an example

$$\vdash (\alpha \to \beta) \vee (\beta \to \alpha)$$

$$I \vdash \alpha \to \beta, \beta \to \alpha$$

by display rules $\quad I < (\alpha \to \beta) \vdash \beta \to \alpha$

$$I < (\alpha \to \beta) \vdash \beta > \alpha$$

and $\quad I < (\beta > \alpha) \vdash \alpha > \beta$

$$I < (\beta > \alpha), \alpha \vdash \beta \qquad \frac{\beta \vdash M}{I < (\beta > \alpha), \alpha \vdash M}$$

Our algorithm: an example

$$\vdash (\alpha \to \beta) \vee (\beta \to \alpha)$$

$$I \vdash \alpha \to \beta, \beta \to \alpha$$

by display rules $\quad I < (\alpha \to \beta) \vdash \beta \to \alpha$

$$I < (\alpha \to \beta) \vdash \beta > \alpha$$

and $\quad I < (\beta > \alpha) \vdash \alpha > \beta$

$$I < (\beta > \alpha), \alpha \vdash \beta \qquad \frac{\beta \vdash M}{I < (\beta > \alpha), \alpha \vdash M}$$

$$\frac{L \vdash \alpha \quad \beta \vdash M \quad Z \vdash \beta \quad \alpha \vdash V}{I \vdash (L > M), (Z > V)} \qquad \frac{L \vdash V \quad Z \vdash M}{I \vdash (L > M), (Z > V)}$$

Further open problems (a selection)

- First-order, modal logics, ...
- ”Applications”:

  E.g.

  - new semantic foundations (e.g. non-deterministic matrices) (AC, O. Lahav, A. Zamansky)
  - automated deduction procedures (AC, E. Pimentel)
  - new algebraic completions
  - admissibility of other rules (e.g. standard completeness)
  - ...