Utrecht University

Structural completeness in logics of dependence

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Joint work with Rosalie Iemhoff

Outline

1. logics of dependence
2. admissible rules and structural completeness

Dependence between first-order variables

First Order Quantifiers:

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \phi$$

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First-order dependence Logic (Väänänen 2007):

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 (=\!(x_2, y_2) \wedge \phi)$$

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$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 (=(x_2, y_2) \wedge \phi)$$

Theorem (Enderton, Walkoe). Over sentences, all of the above extensions of **FO** have the same expressive power as Σ^1_1.

Propositional dependence logic $=$

classical propositional logic $+$ $=(\vec{p}, q)$

- Whether $f(x) > 0$ depends completely on whether $x < 0$ or not.
- I will be absent depending on whether he shows up or not.
- Whether it rains depends completely on whether it is summer or not.

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$$=\!(x, f)$$

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$$=(s, r)$$

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	leap year	summer	rainy
v_1	0	0	1

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This type of dependence corresponds precisely to *functional dependency*, widely investigated in Database Theory.

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Team semantics (Hodges 1997)

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$$v_1 \models =(s, r)?$$

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A *team* X

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v_2	1	0	1
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$$X \models =\!(s, r)$$

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	leap year	summer	rainy
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v_2	1	0	1
v_3	1	1	0
v_4	0	1	0
v_5	1	1	1

Y

$$X \models {=}(s, r) \qquad Y \not\models {=}(s, r)$$

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- Whether it rains depends completely on whether it is summer or not.

Team semantics (Hodges 1997)

A *team* X (rows v_1–v_4), Y (rows v_1–v_5)

	leap year	summer	rainy
v_1	0	0	1
v_2	1	0	1
v_3	1	1	0
v_4	0	1	0
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$$X \models {=}(s, r), \qquad Y \not\models {=}(s, r)$$

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Logics of dependence

- Well-formed formulas of *propositional dependence logic* (PD) are given by the following grammar

$$\phi ::= \; p \mid \neg p \mid =(\vec{p}, q) \mid \phi \wedge \phi \mid \phi \vee \phi$$

- *propositional intuitionistic dependence logic* (PID):

$$\phi ::= p \mid \bot \mid =(\vec{p}, q) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi$$

	p_0	p_1	p_2	...
v_1	1	0	0	...

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A valuation is a function $v : \mathrm{Prop} \to \{0, 1\}$.

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A valuation is a function $v : \mathrm{Prop} \to \{0, 1\}$.
A *team* is a set of valuations.

	p_0	p_1	p_2	...
v_1	1	0	0	...
v_2	1	1	0	...
v_3	0	1	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Team Semantics

Let X be a team.

- $X \models p$ iff for all $v \in X$, $v(p) = 1$;
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	p	q	r
v_1	1	0	0
v_2	1	0	1
v_3	0	1	0
v_4	0	1	1

A formula ϕ is said to be flat iff for all teams X,

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$X \models p$ $\qquad X \cup Y \not\models p$

$Y \models \neg p$ $\qquad X \cup Y \not\models \neg p$

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$$X \models \phi \iff \forall v \in X: \{v\} \models \phi$$

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$Y \models \phi$

$Z \models \psi$

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Consider the algebra $(\mathcal{L}(\wp(2^N)), \otimes, \cap, \cup, \{\emptyset\}, \subseteq)$, where $A \otimes B = \downarrow \{X \cup Y \mid X \in A \text{ and } Y \in B\}$.

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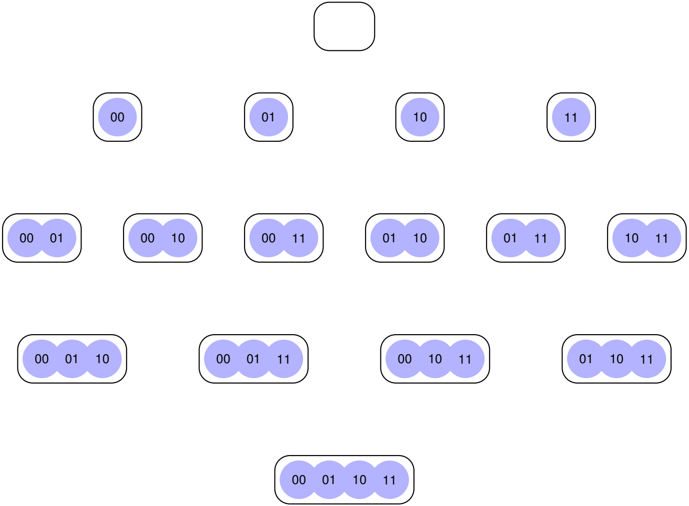

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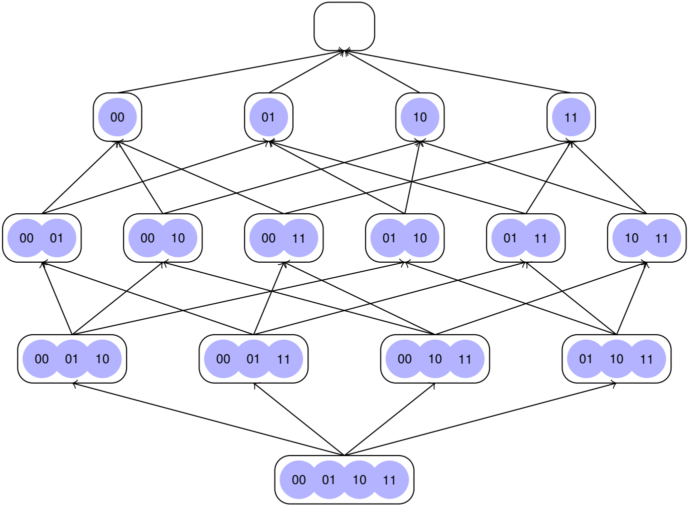

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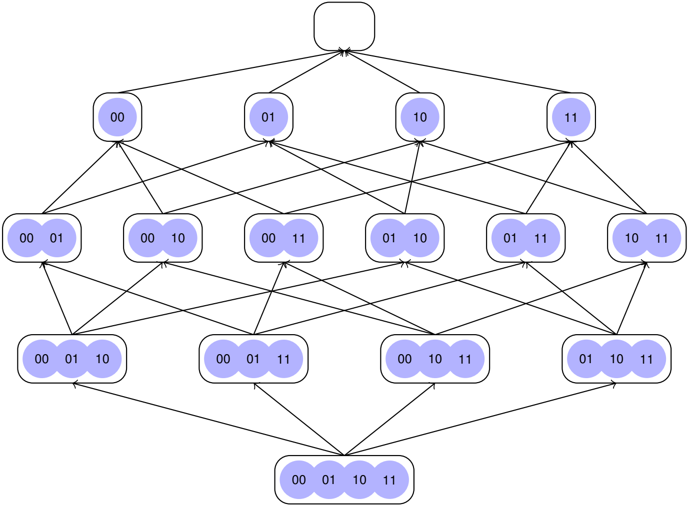

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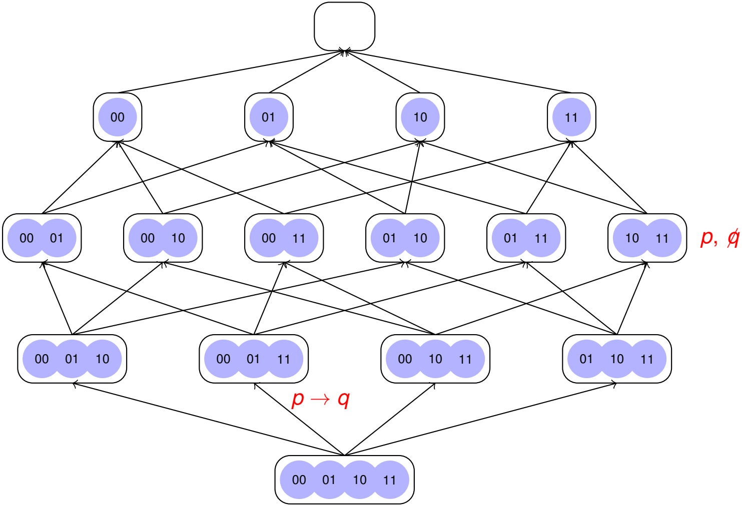

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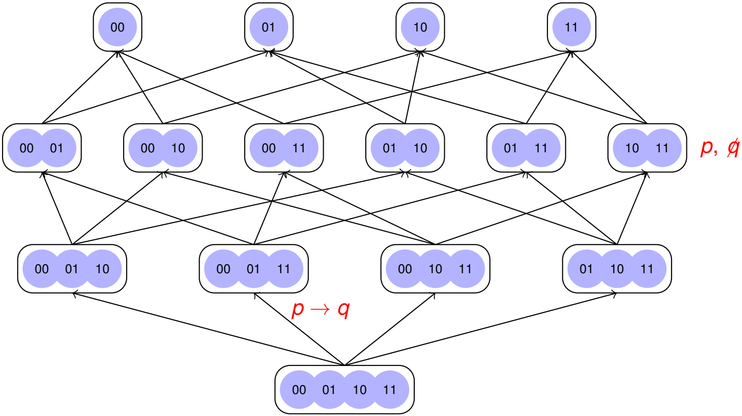

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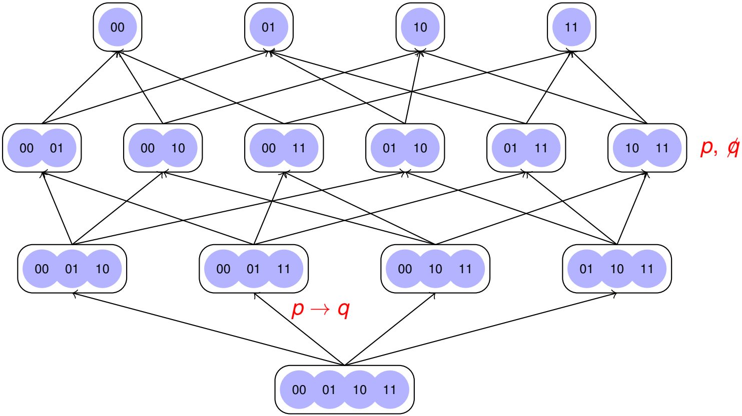

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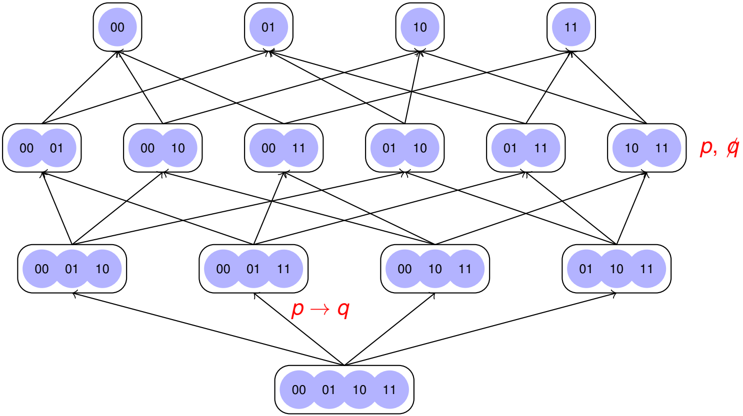

(Ciardelli and Roelofsen, 2011):

[Recall: **ND** $\subseteq$ **KP** $\subseteq$ **ML**]

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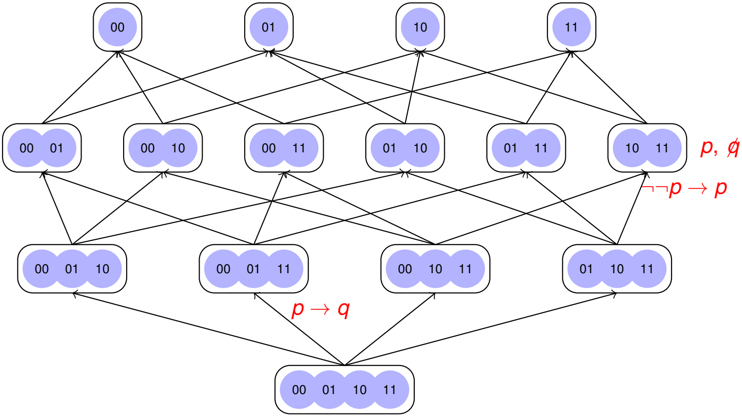

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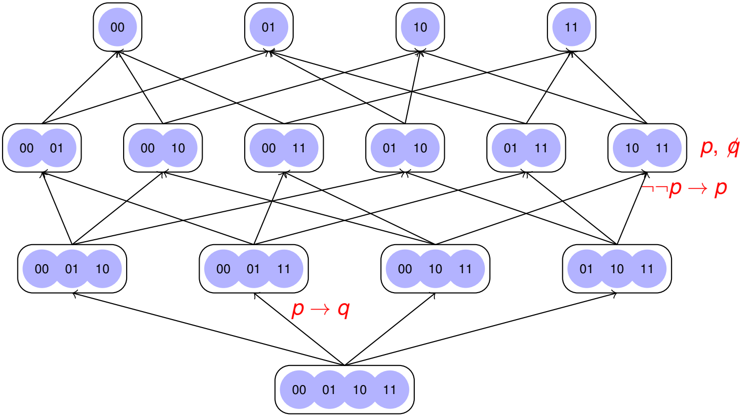

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Axioms:

- *all substitution instances of* **IPC** *axioms*
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One can expand the languages of PD and PID such that for all flat formulas ϕ and ψ, strings of the form $=\!(\phi, \psi)$, $\neg\phi$ are well-formed formulas. There are sound and complete deductive systems for the extended logics **PD** and **PID**.

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admissible rules and structural completeness

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- A logic L is a set of theorems, i.e., $\mathsf{L} = \{\phi : \emptyset \vdash_{\mathsf{L}} \phi\}$.

- A rule ϕ/ψ of L is said to be *admissible*, in symbols $\phi \mid\!\sim_{\mathsf{L}} \psi$, if $\vdash_{\mathsf{L}} \sigma(\phi) \Longrightarrow \vdash_{\mathsf{L}} \sigma(\psi)$ for all substitutions σ.
- Alternatively, a rule *R* is admissible in L iff $\{\phi : \emptyset \vdash_{\mathsf{L}} \phi\} = \{\phi : \emptyset \vdash_{\mathsf{L}}^{R} \phi\}$.

[Friedman, Citkin, Rybakov, Ghilardi, etc.]

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  Pf. For any σ,

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A logic L is said to be *$\mathcal{S}$-structurally complete* if every $\mathcal{S}$-admissible rule is derivable in L, i.e., $\phi \mathrel{|\!\sim}^{\mathcal{S}}_{\mathsf{L}} \psi \iff \phi \vdash_{\mathsf{L}} \psi$.

Example:

- KP rule $\neg p \to q \vee r / (\neg p \to q) \vee (\neg p \to r)$ is admissible in all intermediate logics, but KP rule is not derivable in **IPC**.
- **KP** is not structurally complete, **ML** is structurally complete.
- **CPC** is structurally complete.

Theorem

PD *and* **PID** *are $\mathcal{F}$-structurally complete, where $\mathcal{F}$ is the class of all flat substitutions.*

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$\mathbf{ND}^{\neg}$, $\mathbf{KP}^{\neg}$ *and* $\mathbf{ML}^{\neg}$ *are $\mathcal{ST}$-structurally complete, where $\mathcal{ST}$ is the class of all stable substitutions, i.e., substitutions σ s.t. $\vdash \neg\neg\sigma(p) \leftrightarrow \sigma(p)$.*

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Lemma

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Proof.

$$X \left\{ \begin{array}{c|cc} & p & q \\ \hline v_1 & 1 & 1 \\ v_2 & 1 & 0 \\ v_3 & 0 & 1 \end{array} \right.$$

Let

$$\Theta_X := \begin{cases} & \text{for } \mathbf{PD}; \\ & \text{for } \mathbf{PID}. \end{cases}$$

Then $Y \models \Theta_X \iff Y \subseteq X$, for any team Y on N.

Corollary

$\phi \equiv \bigvee_{X \in [\![\phi]\!]} \Theta_X$, where $[\![\phi]\!] = \{X \subseteq \{0,1\}^V \mid X \models \phi\}$, for any consistent formula ϕ of **PD** *and* **PID**.

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Definition (Projective formula)

Let $\mathcal{S}$ be a set of substitutions under which $\vdash_{\mathsf{L}}$ is closed. A formula ϕ is said to be *$\mathcal{S}$-projective* in L if there exists $\sigma \in \mathcal{S}$ such that

(1) $\vdash_{\mathsf{L}} \sigma(\phi)$

(2) $\phi, \sigma(\psi) \vdash_{\mathsf{L}} \psi$ and $\phi, \psi \vdash_{\mathsf{L}} \sigma(\psi)$ for all formulas ψ.

Such σ is called a $\mathcal{S}$-projective unifier of ϕ in L.

- Every consistent formula is projective in **CPC**.
- Every consistent negated formula (i.e. $\neg\alpha$) is projective in every intermediate logic.
- For $\mathsf{L} \in \{\mathbf{PD}, \mathbf{PID}\}$, the formula

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- Every consistent negated formula (i.e. $\neg\phi$) is projective in every intermediate logic. Moreover, every consistent $\neg\phi$ is projective in $\mathsf{L}^\neg$, where L is an intermediate logic s.t. ND $\subseteq$ L.
- For L $\in \{\mathbf{PD}, \mathbf{PID}\}$, the formula

$$\Theta_X = \begin{cases} \bigotimes_{v \in X} (p_1^{v(p_1)} \wedge \cdots \wedge p_n^{v(p_n)}), & \text{for } \mathbf{PD}; \\ \neg\neg \bigvee_{v \in X} (p_1^{v(p_1)} \wedge \cdots \wedge p_n^{v(p_n)}), & \text{for } \mathbf{PID}. \end{cases}$$

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Theorem

$\mathsf{L} \in \{\mathbf{PD}, \mathbf{PID}\}$ *is $\mathcal{F}$-structurally complete, i.e.,* $\phi \mid\!\sim^{\mathcal{F}}_{\mathsf{L}} \psi \iff \phi \vdash_{\mathsf{L}} \psi$.

Recall: $\phi \equiv \bigvee_{i \in I} \Theta_{X_i}$

Example

Let $\mathsf{L} \in \{\mathbf{PD}, \mathbf{PID}\}$. If $\Theta_X \mid\!\sim^{\mathcal{F}} \psi$, then $\Theta_X \vdash_{\mathsf{L}} \psi$

Proof. Let $\sigma \in \mathcal{F}$ be a projective unifier of Θ_X. Then $\vdash \sigma(\Theta_X)$. Now, since $\Theta_X \mid\!\sim^{\mathcal{F}}_{\mathsf{L}} \psi$, we obtain that $\vdash \sigma(\psi)$.

On the other hand, as σ is a projective unifier of Θ_X, we have that $\Theta_X, \sigma(\psi) \vdash \psi$, thus $\Theta_X \vdash \psi$ for all $i \in I$, as desired. □

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For any intermediate logic L *such that* ND $\subseteq$ L*, its negative variant* $\mathsf{L}^{\neg} = \{\phi \mid \tau(\phi) \in \mathsf{L},\ \textit{where}\ \tau(p) = \neg p\}$ *is $\mathcal{ST}$-hereditarily structurally complete, i.e.,* L′ *is $\mathcal{ST}$-structurally complete, for any intermediate theory* L′ *extending* L *such that* $\vdash_{\mathsf{L}'}$ *is closed under $\mathcal{ST}$.*

In particular, **ND**$^{\neg}$, **KP**$^{\neg}$ *and* **ML**$^{\neg}$ *are $\mathcal{ST}$-hereditarily structurally complete.*

- **ML** is hereditarily structurally complete.
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