The Canonical FEP Construction for Residuated Lattice-Ordered Algebras

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The FEP

Definition

A class $\mathcal{K}$ of algebras has the finite embeddability property (FEP, for short) if:

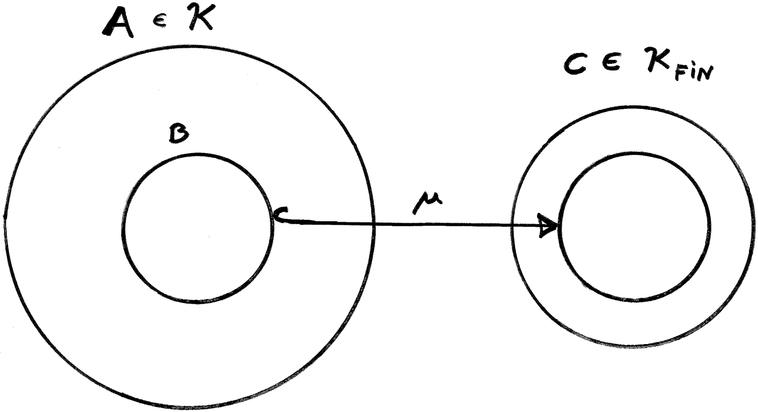
for any $\boldsymbol{A} \in \mathcal{K}$ and any *finite* $B \subseteq A$,

- there exists a *finite* algebra $\boldsymbol{C} \in \mathcal{K}$ and
- an embedding $\mu : B \to \boldsymbol{C}$ such that
- all existing operations in B are preserved.

E.g., if $a, b \in B$ and $a \circ^{\boldsymbol{A}} b \in B$, then $\mu(a \circ^{\boldsymbol{A}} b) = \mu(a) \circ^{\boldsymbol{C}} \mu(b)$.

The 'partial subalgebra' B is isomorphically embeddable into $\boldsymbol{C}$.

Schematic of the FEP



On the FEP

The FEP for a class of algebras $\mathcal{K}$ implies the following:

- $\mathcal{K}$ has the finite model property in the sense that any identity that fails in $\mathcal{K}$ will fail in a finite algebra in $\mathcal{K}$. (In fact, the strong finite model property.)
- If $\mathcal{K}$ is a (quasi)variety with the FEP then $\mathcal{K}$ is generated by its finite members.
- If $\mathcal{K}$ is finitely axiomatized, then its universal theory is decidable.

On the FEP

The FEP for Integral (commutative) Residuated Lattices was proved in:

- Blok, VA: The finite embeddability property for residuated lattices, pocrims and BCK-algebras (2002)

The construction used in the above paper is a form of the MacNeille completion. The details are in:

- VA: Completion and finite embeddability property for residuated ordered algebras (2009).

But see also:

- Galatos, Jipsen: Residuated frames with applications to decidability (2012)
- Ciabattoni, Galatos, Terui: From axioms to analytic rules in nonclassical logics (2008)
- Belardinelli, Jipsen, Ono: Algebraic aspects of cut elimination (2004)
- Ono: Closure operators and complete embeddings of residuated lattices (2003)

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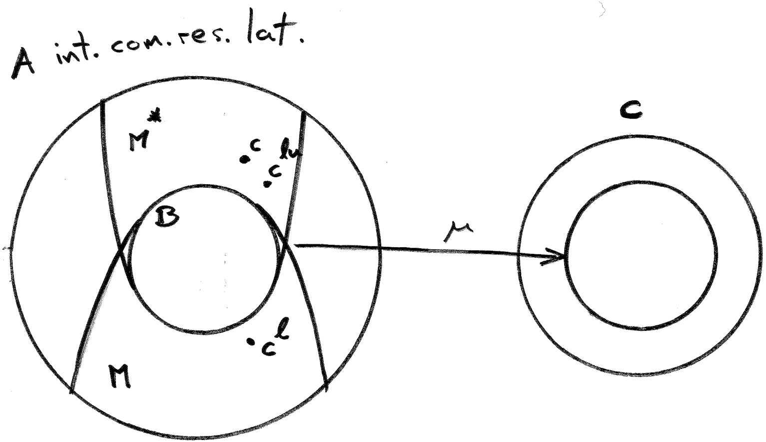
MacNeille FEP construction

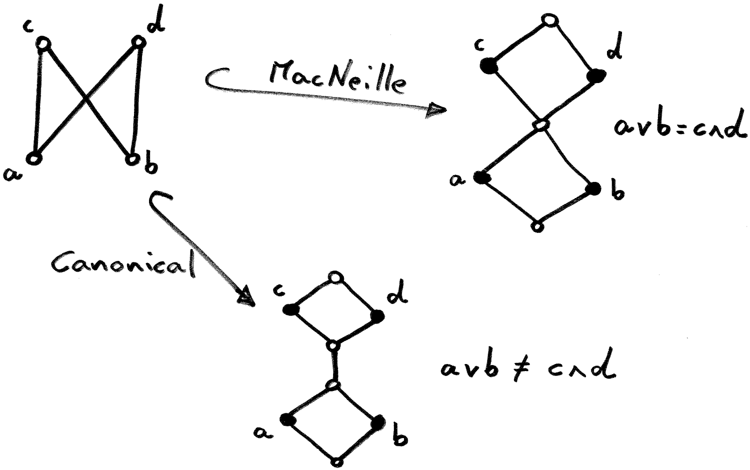


Problem:

Is there a canonical extension version of the FEP for residuated lattice type algebras?

If so, what properties of the canonical extension hold in this context?

Difference between MacNeille Completion and Canonical Extension



Canonical Extension for Posets

Given a poset $\boldsymbol{P}$, set

$\mathcal{F}$: the set of filters of $\boldsymbol{P}$

$\mathcal{I}$: the set of ideals of $\boldsymbol{P}$

Define: $R \subseteq \mathcal{F} \times \mathcal{I}$ by $\quad (F, I) \in R \;\Leftrightarrow\; F \cap I \neq \emptyset$

For $X \subseteq \mathcal{I}$ and $\Lambda \subseteq \mathcal{F}$, define:

$$X^{\triangleleft} = \{F \in \mathcal{F} : (F, I) \in R \text{ for all } I \in \mathcal{I}\}$$

$$\Lambda^{\triangleright} = \{I \in \mathcal{I} : (F, I) \in R \text{ for all } F \in \Lambda\}$$

Then $X \subseteq \mathcal{I}$ is Galois closed if $X^{\triangleleft\triangleright} = X$

(The 'polarities' $^{\triangleleft}$ and $^{\triangleright}$ form a Galois connection.)

Canonical Extension for Posets

- Let P^c be the set of all Galois closed subsets of $\mathcal{I}$.
- Then $\boldsymbol{P}^c = \langle P^c, \supseteq \rangle$ is a complete lattice, called the canonical extension of $\boldsymbol{P}$.
- The map:

$$\mu(a) = \{I \in \mathcal{I} : a \in I\}$$

is an embedding of $\boldsymbol{P}$ into $\boldsymbol{P}^c$ preserving all existing finite meets and joins.

- Dunn, Gehrke, Palmigiano: Canonical Extensions and Relational Completeness of Some Substructural Logics (2005)
- Galatos, Jipsen, Kowalski, Ono: Residuated Lattices: An Algebraic Glimpse at Substructural Logics (2007)
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Canonical Extension for Posets

Thm: [Morton 2014]
The canonical extension of a poset is internally compact, i.e.,

For any $S, T \subseteq P$, $\bigwedge \mu[S] \leq \bigvee \mu[T]$ if, and only if,
$(F, I) \in R$ for any $F \in \mathcal{F}$ s.t. $S \subseteq F$ and any $I \in \mathcal{I}$ s.t. $T \subseteq I$.

Algebraic Structures

Given a poset $\langle P, \leq \rangle$:

A unary operation f on P is residuated if there exists a unary 'residual' function g on A such that, for all $a, b, \in A$,

$$f(a) \leq b \;\Leftrightarrow\; a \leq g(b).$$

A binary operation $\circ$ on P is residuated if there exist binary 'residual' functions $\backslash, /$ on A such that, for all $a, b, c \in A$,

$$a \circ b \leq c \;\Leftrightarrow\; b \leq a \backslash c \;\Leftrightarrow\; a \leq c/a.$$

We say:

f is decreasing if $f(a) \leq a$ for all $a \in P$,

$\circ$ is decreasing if $a \circ b \leq a$ and $a \circ b \leq b$ for all $a, b \in P$.

Algebraic Structures

A Residuated Lattice-Ordered Algebra is of the form:

$$\boldsymbol{A} = \langle A, \wedge, \vee, \mathbb{T}, \mathbb{T}^* \rangle$$

where

- $\langle A, \wedge, \vee \rangle$ is a lattice,
- $\mathbb{T}$ consists of a finite set of residuated unary and binary operations and constants, and
- $\mathbb{T}^*$ consists of the residuals of all operations in $\mathbb{T}$.

We call $\boldsymbol{A}$ decreasing if all $f, \circ \in \mathbb{T}$ are decreasing.

Canonical FEP Construction

$\mathbf{A} = \langle A, \wedge, \vee, \circ, f, \backslash, /, g \rangle$ a residuated lattice-ordered algebra

$B \subseteq_{fin} A$.

Define:

M: the closure of B under $\circ, f, \wedge$

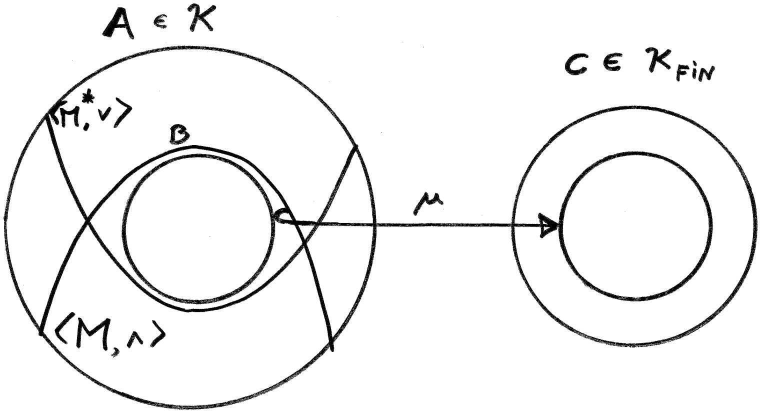
M^*: the closure of B under $\vee, g$ and $\{a \backslash x : a \in M\} \cup \{x / a : a \in M\}$

Note:

$\langle M, \wedge \rangle$ is a meet-semilattice and

$\langle M^*, \vee \rangle$ is a join-semilattice.

Schematic of Canonical FEP Construction



Canonical FEP Construction

Define:

$\mathcal{F}(M)$: the set of filters of $\langle M, \wedge \rangle$

$\mathcal{I}(M^*)$: the set of ideals of $\langle M^*, \vee \rangle$

Define: $R \subseteq \mathcal{F}(M) \times \mathcal{I}(M^*)$ by:

$$(F, I) \in R \quad \Leftrightarrow \quad (\exists a \in F)(\exists b \in I)[a \leq b]$$

For $X \subseteq \mathcal{I}(M^*)$ and $\Lambda \subseteq \mathcal{F}(M)$, define:

$$X^{\triangleleft} = \{F \in \mathcal{F}(M) : (F, I) \in R \text{ for all } I \in X\}$$

$$\Lambda^{\triangleright} = \{I \in \mathcal{I}(M^*) : (F, I) \in R \text{ for all } F \in \Lambda\}$$

Then $X \subseteq \mathcal{I}(M^*)$ is Galois closed if $X^{\triangleleft\triangleright} = X$

Canonical FEP construction

Let C be the set of Galois closed subsets of $\mathcal{I}(M^*)$.

For $X, Y \in C$, define:

$$X \vee^C Y = X \cap Y \quad \text{and} \quad X \wedge^C Y = (X \cup Y)^{\lhd\rhd}$$

Then: $\langle C, \wedge^C, \vee^C \rangle$ is a complete lattice.

Note: If we take B to be the whole of A, then C is the canonical extension of A.

Canonical FEP construction

Next, we need to define operations f^C, $\circ^C$, g^C, $\backslash^C$ and $/^C$ on C.

For $X \in C$, define:

$$f^C(X) = f(X^{\triangleleft})^{\triangleright}$$

here $f(X^{\triangleleft}) = \{[f(F)\rangle : F \in X^{\triangleleft}\}$

and $f(F) = \{f(a) : a \in F\}$.

For $X, Y \in C$, define:

$$X \circ^C Y = (X^{\triangleleft} \circ Y^{\triangleleft})^{\triangleright}$$

and

$$g^C(X) = g(X)^{\triangleleft\triangleright}$$

$$X \backslash^C Y = (X^{\triangleleft} \backslash Y)^{\triangleleft\triangleright}$$

$$Y /^C X = (Y / X^{\triangleleft})^{\triangleleft\triangleright}$$

Canonical FEP construction

Thm:
$\boldsymbol{C} = \langle C, \wedge^C, \vee^C, \circ^C, f^C, \backslash^C, /^C, g^C \rangle$ is a decreasing residuated lattice-ordered algebra.

Let $\mu : B \to C$ be defined by

$$\mu(b) = \{I \in \mathcal{I}(M^*) : b \in I\}.$$

Thm:
μ is an embedding that preserves all existing operations in B.

Canonical FEP construction

Thm: C is finite.

The proof uses the following:

- $\langle M, \leq \rangle$ is reverse well-quasi-ordered,
- $\langle M^*, \leq \rangle$ is well-quasi-ordered

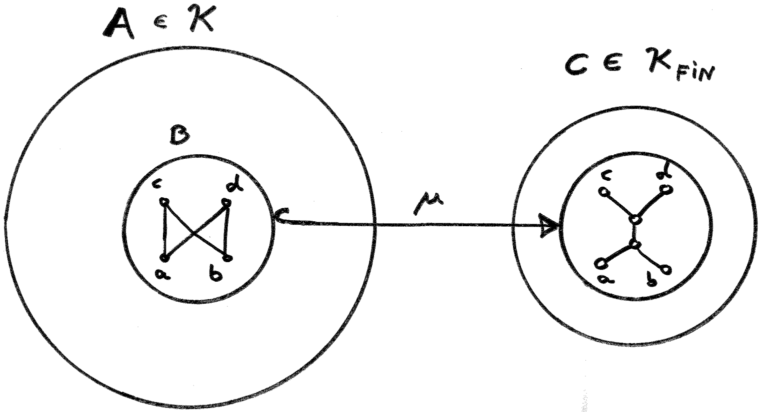
and relies heavily on the theory of well-quasi-orders, esp:

Higman: Ordering by divisibility in abstract algebras (1952)

Nash-Williams: On well-quasi-ordering finite trees (1963)

Thm: Classes of decreasing residuated lattice-ordered algebras have the FEP via a canonical FEP construction.

Internal Compactness



σ- and π- extensions

Let $\boldsymbol{P}$ be a poset and $\boldsymbol{P}^c$ its canonical extension with embedding $\mu : P \to P^c$.
An element of P^c is closed if it's of the form:

$$\bigwedge \mu[F], \quad \text{where } F \text{ is a filter of } \boldsymbol{P}.$$

An element of P^c is open if it's of the form:

$$\bigvee \mu[I], \quad \text{where } I \text{ is an ideal of } \boldsymbol{P}.$$

Note: every element of P^c is a join of closed elements and also a meet of open elements.

σ- and π- extensions

Let $h : P \to P$. Then $h^\sigma, h^\pi : P^c \to P^c$ are defined by:

$$h^\sigma(X) = \bigvee\{\bigwedge\{\mu(h(a)) : a \in P,\ Y \leq \mu(a)\} : Y \text{ closed } \& \ X \geq Y\}$$

$$h^\pi(X) = \bigwedge\{\bigvee\{\mu(h(a)) : a \in P,\ \mu(a) \leq Z\} : Z \text{ open } \& \ X \leq Z\}.$$

Note: In the canonical FEP construction, we don't have a description of closed and open elements.

Thm: For any order-preserving operation h on P, and $X \in P^c$,

$$h^\sigma(X) = h(X^{\lhd})^{\rhd} \quad \text{and} \quad h^\pi(X) = h(X)^{\lhd\rhd}$$

- For $\circ, f$, the extensions $\circ^C$ and f^C are σ-extensions and
- for $\backslash, /, g$, the extensions $\backslash^C, /^C, g^C$ are π-extensions.

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Preservation of Properties

We say a property is preserved by the canonical FEP construction if, whenever $\boldsymbol{A}$ satisfies the property, then $\boldsymbol{C}$ does as well.

Thm: The following properties are preserved:

- the decreasing property for $\circ$ and f
- associativity, commutativity or idempotence of $\circ$
- identity ($x \circ 1 = x$)
- upper and lower bounds

Preservation of Properties

Given an inequality $s \leq t$, is there a syntactic way of deciding if the property is preserved?

i.e., if $(\forall \vec{a})(s^{\boldsymbol{A}}(\vec{a}) \leq t^{\boldsymbol{A}}(\vec{a}))$, do we have $(\forall \vec{X})(s^{\boldsymbol{C}}(\vec{C}) \leq t^{\boldsymbol{C}}(\vec{C}))$?

For any term s and elements $\vec{X} \in C$ we define approximations to $s^{\boldsymbol{C}}(\vec{X})$ using the σ- and π-extensions.

Following

Jonsson: On the canonicity of Sahlqvist identities (1994)

and using the result on σ- and π-extensions, we define expanding and contracting terms that allow us to obtain preservation results.

Preservation of Properties

Thm:
The canonical FEP construction preserves inequalities of the form $s \leq t$, where

- s is a term in the language $\mathbb{T}$ acting on finite meets of variables, where any $\circ$ acts on terms with no common variables,
- t is any term in the language $\mathbb{T} \cup \{\wedge\}$.

Thm:
Any class of decreasing residuated lattice-ordered algebras axiomatized by a set of inequalities of the above type has the FEP.

THE END