# The theory of topos-theoretic *bridges*, five years later

Olivia Caramello

Université Paris 7

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# Toposes as unifying 'bridges' in Mathematics

In this lecture, whenever I use the word 'topos', I really mean 'Grothendieck topos'.

The theory of topos-theoretic 'bridges' was introduced in the paper

*The unification of Mathematics via Topos Theory*

in 2010.

This theory provides means for exploiting the technical flexibility inherent to the concept of topos to build unifying 'bridges' across different mathematical theories having an equivalent, or strictly related, semantic content.

In the past five years, many applications of this general methodology have been obtained in different fields of Mathematics. In fact, 'bridges' have proved useful not only for connecting different theories with each other, but also for working inside a fixed mathematical domain.

# A few selected applications

- Model theory (topos-theoretic Fraïssé theorem)
- Proof theory (various results for first-order theories)
- Algebra (topos-theoretic generalization of topological Galois theory)
- Topology (topos-theoretic interpretation/generation of Stone-type and Priestley-type dualities)
- Functional analysis (various results on Gelfand spectra and Wallman compactifications)
- Many-valued logics and lattice-ordered groups (two joint papers with A. C. Russo)
- Cyclic homology, as reinterpreted by A. Connes (work on *"cyclic theories"*, jointly with N. Wentzlaff)
- Algebraic geometry (logical analysis of (co)homological motives, cf. the paper *"Syntactic categories for Nori motives"* joint with L. Barbieri-Viale and L. Lafforgue)

# Plan of the talk

- Topos-theoretic background
- The 'bridge-building' technique: its key principles and the underlying vision
- Analysis of a few notable 'bridges' in light of the general theory
- Future perspectives and the unification programme

# The eclectic nature of toposes

Toposes are particularly eclectic objects, which can be profitably approached from different points of view.

In fact, as it is well-known, a Grothendieck topos can be seen as:

- a generalized space
- a mathematical universe
- a theory modulo 'Morita-equivalence'

We shall now briefly review each of these different points of view.

# Toposes as generalized spaces

- The notion of topos was introduced in the early sixties by A. Grothendieck with the aim of bringing a topological or geometric intuition also in areas where actual topological spaces do not occur.
- Grothendieck realized that many important properties of topological spaces $X$ can be naturally formulated as (invariant) properties of the categories $\mathbf{Sh}(X)$ of sheaves of sets on the spaces.
- He then defined toposes as more general categories of sheaves of sets, by replacing the topological space $X$ by a pair $(\mathscr{C}, J)$ consisting of a (small) category $\mathscr{C}$ and a 'generalized notion of covering' $J$ on it, and taking sheaves (in a generalized sense) over the pair:

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, squiggly] & \mathbf{Sh}(X) \arrow[d, squiggly] \\ (\mathscr{C}, J) \arrow[r, dashed] & \mathbf{Sh}(\mathscr{C}, J) \end{tikzcd}$$

# Toposes as mathematical universes

A decade later, W. Lawvere and M. Tierney discovered that a topos could not only be seen as a generalized space, but also as a mathematical universe in which one can do mathematics similarly to how one does it in the classical context of sets (with the only exception that one must argue constructively).

Amongst other things, this discovery made it possible to:

- Exploit the inherent 'flexibility' of the notion of topos to construct 'new mathematical worlds' having particular properties.
- Consider models of any kind of (first-order) mathematical theory not just in the classical set-theoretic setting, but inside every topos, and hence 'relativise' Mathematics.

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# Toposes as theories up to 'Morita-equivalence'

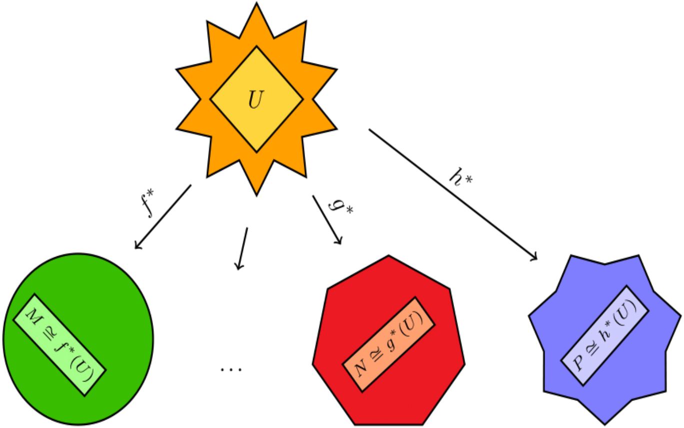
It was realized in the seventies (thanks to the work of several people, notably including W. Lawvere, A. Joyal, G. Reyes and M. Makkai) that:

- To any (geometric first-order) mathematical theory $\mathbb{T}$ one can canonically associate a topos $\mathscr{E}_{\mathbb{T}}$, called the classifying topos of the theory, which represents its 'semantical core'.
- The topos $\mathscr{E}_{\mathbb{T}}$ is characterized by the following universal property: for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories

  $$\textbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

  *natural* in $\mathscr{E}$, where $\textbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}})$ is the category of geometric morphisms $\mathscr{E} \to \mathscr{E}_{\mathbb{T}}$ and $\mathbb{T}\text{-mod}(\mathscr{E})$ is the category of $\mathbb{T}$-models in $\mathscr{E}$.

# Toposes as theories up to 'Morita-equivalence'




Classifying topos

# Toposes as theories up to 'Morita-equivalence'

- Two mathematical theories have the same classifying topos (up to equivalence) if and only if they have the same 'semantical core', that is if and only if they are indistinguishable from a semantic point of view; such theories are said to be Morita-equivalent.
- Conversely, every Grothendieck topos arises as the classifying topos of some theory.
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# Toposes as *bridges*

- The notion of Morita-equivalence is ubiquitous in Mathematics; indeed, it formalizes in many situations the feeling of 'looking at the same thing in different ways', or 'constructing a mathematical object through different methods'.
- In fact, many important dualities and equivalences in Mathematics can be naturally interpreted in terms of Morita-equivalences.
- On the other hand, Topos Theory itself is a primary source of Morita-equivalences. Indeed, different representations of the same topos can be interpreted as Morita-equivalences between different mathematical theories.
- Any two theories which are biinterpretable in each other are Morita-equivalent but, very importantly, the converse does not hold.
- Moreover, the notion of Morita-equivalence captures the intrinsic dynamism inherent to the notion of mathematical theory; indeed, a mathematical theory alone gives rise to an infinite number of Morita-equivalences.

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# Toposes as *bridges*

- The existence of different theories with the same classifying topos translates, at the technical level, into the existence of different representations (technically speaking, sites of definition) for the same topos.

- Topos-theoretic invariants can thus be used to transfer information from one theory to another:

$$
\begin{tikzcd}
 & \mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{T}'} \arrow[dr, dashed] & \\
\mathbb{T} \arrow[ur, dashed] & & \mathbb{T}'
\end{tikzcd}
$$

- The transfer of information takes place by expressing a given invariant in terms of the different representation of the topos.

# *Toposes as bridges*

- As such, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.
- Any topos-theoretic invariant behaves in this context like a 'pair of glasses' which allows to discern certain information which is 'hidden' in the given Morita-equivalence; different invariants allow to transfer different information.
- This methodology is technically effective because the relationship between a topos and its representations is often very natural, enabling us to easily transfer invariants across different representations (and hence, between different theories).
- The level of generality represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories. Indeed, as shown in my papers, important topos-theoretic invariants considered on the classifying topos $\mathscr{E}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ translate into interesting logical (i.e. syntactic or semantic) properties of $\mathbb{T}$.

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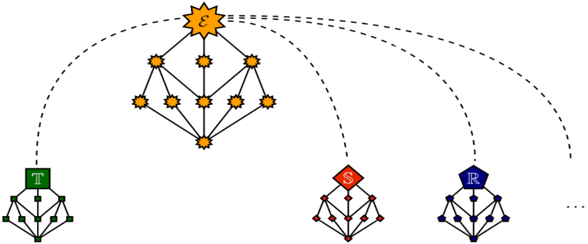
# Toposes as bridges

- The fact that topos-theoretic invariants specialize to important properties or constructions of natural mathematical interest is a clear indication of the centrality of these concepts in Mathematics. In fact, whatever happens at the level of toposes has 'uniform' ramifications in Mathematics as a whole: for instance

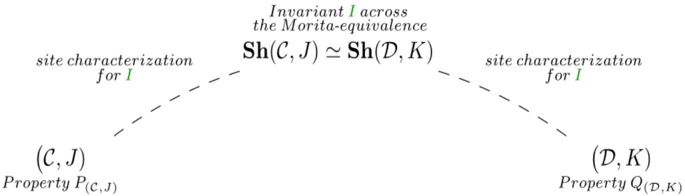



Lattices of theories

This picture represents the lattice structure on the collection of the subtoposes of a topos $\mathscr{E}$ inducing lattice structures on the collection of 'quotients' of geometric theories $\mathbb{T}$, $\mathbb{S}$, $\mathbb{R}$ classified by it.

# The 'bridge-building' technique

- Decks of 'bridges': Morita-equivalences (or more generally morphisms or other kinds of relations between toposes)
- Arches of 'bridges': Site characterizations (or more generally 'unravelings' of topos-theoretic invariants in terms of concrete representations of the relevant topos)




The 'bridge' yields a logical equivalence (or an implication) between the 'concrete' properties $P_{(\mathscr{C},J)}$ and $Q_{(\mathscr{D},K)}$, interpreted in this context as manifestations of a unique property $I$ lying at the level of the topos.

# Toposes as 'bridges' and the Erlangen Program

*The methodology 'toposes as bridges' is a vast extension of Felix Klein's Erlangen Program (A. Joyal)*

More specifically:

- Every group gives rise to a topos (namely, the category of actions on it), but the notion of topos is much more general.
- As Klein classified geometries by means of their automorphism groups, so we can study first-order geometric theories by studying the associated classifying toposes.
- As Klein established surprising connections between very different-looking geometries through the study of the algebraic properties of the associated automorphism groups, so the methodology 'toposes as bridges' allows to discover non-trivial connections between properties, concepts and results pertaining to different mathematical theories through the study of the categorical invariants of their classifying toposes.

# Structural translations

The method of bridges can be interpreted linguistically as a methodology for translating concepts from one context to another.

But which kind of translation is this?

In general, we can distinguish between two essentially different approaches to translation.

- The 'dictionary-oriented' or 'bottom-up' approach, consisting in a dictionary-based renaming of the single words composing the sentences.
- The 'invariant-oriented' or 'top-down' approach, consisting in the identification of appropriate concepts that should remain invariant in the translation, and in the subsequent analysis of how these invariants can be expressed in the two languages.

The topos-theoretic translations are of the latter kind. Indeed, the invariant properties are topos-theoretic invariants defined on toposes, and the expression of these invariants in terms of the two different theories is essentially determined by the structural relationship between the topos and its two different representations.

# Some examples of 'bridges'

We shall now discuss a few 'bridges' established in the context of the applications mentioned at the beginning of the talk:

- Theories of presheaf type
- Topos-theoretic Fraïssé theorem
- Topological Galois theory
- Stone-type dualities

The results are completely *different*... but the methodology is always the *same*!

# Theories of presheaf type

## Definition

A geometric theory is said to be of presheaf type if it is classified by a presheaf topos.

Theories of presheaf type are very important in that they constitute the basic 'building blocks' from which every geometric theory can be built. Indeed, as every Grothendieck topos is a subtopos of a presheaf topos, so every geometric theory is a 'quotient' of a theory of presheaf type.

Every finitary algebraic theory is of presheaf type, but this class contains many other interesting mathematical theories.

Any theory of presheaf type $\mathbb{T}$ gives rise to two different representations of its classifying topos, which can be used to build 'bridges' connecting its syntax and semantics:

$$\begin{tikzcd} & {[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ \text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} & & (\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}}) \end{tikzcd}$$

Here $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ denotes the category of finitely presentable $\mathbb{T}$-models and $(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the syntactic site of $\mathbb{T}$.

# Theories of presheaf type

Here are two examples of theorems obtained by applying the 'bridge' technique:

## Theorem

*Let $\mathbb{T}$ be a theory of presheaf type over a signature $\Sigma$. Then*

(i) *Any finitely presentable $\mathbb{T}$-model in* **Set** *is presented by a $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over $\Sigma$;*

(ii) *Conversely, any $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over $\Sigma$ presents a $\mathbb{T}$-model.*

*In fact, the category $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op}$ is equivalent to the full subcategory $\mathscr{C}_{\mathbb{T}}^{irr}$ of $\mathscr{C}_{\mathbb{T}}$ on the $\mathbb{T}$-irreducible formulae.*

*Irreducible object*

$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}$

*Every object*

$(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

$\mathbb{T}-$*irreducible formula*

# Theories of presheaf type

## Theorem

*Let $\mathbb{T}$ be a theory of preshef type and suppose that we are given, for every finitely presentable* **Set***-model $\mathcal{M}$ of $\mathbb{T}$, a subset $R_{\mathcal{M}}$ of $\mathcal{M}^n$ in such a way that every $\mathbb{T}$-model homomorphism $h : \mathcal{M} \to \mathcal{N}$ maps $R_{\mathcal{M}}$ into $R_{\mathcal{N}}$. Then there exists a geometric formula-in-context $\phi(x_1, \ldots, x_n)$ such that $R_{\mathcal{M}} = [[\vec{x} \, . \, \phi]]_{\mathcal{M}}$ for each finitely presentable $\mathbb{T}$-model $\mathcal{M}$.*

*Subobject of $UA_1 \times \cdots \times UA_n$*

$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}$

*Functorial assignment*

$M \to R_M \subseteq MA_1 \times \cdots \times MA_n$

$(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

*Geometric formula*

$\phi(x_1^{A_1}, \ldots, x_n^{A_n})$

# Topos-theoretic Fraïssé theorem

The following result, which generalizes Fraïssé's theorem in classical model theory, arises from a triple 'bridge'.

## Definition

A set-base model $M$ of a geometric theory $\mathbb{T}$ is said to be homogeneous if for any arrow $y : c \to M$ in $\mathbb{T}\text{-mod}(\mathbf{Set})$ and any arrow $f$ in $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ there exists an arrow $u$ in $\mathbb{T}\text{-mod}(\mathbf{Set})$ such that $u \circ f = y$:

$$\begin{tikzcd} c \arrow[r, "y"] \arrow[d, "f"'] & M \\ d \arrow[ur, dashed, "u"'] & \end{tikzcd}$$

## Theorem

*Let $\mathbb{T}$ be a theory of presheaf type such that the category $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ is non-empty and has AP and JEP. Then the theory $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models is complete and atomic; in particular, assuming the axiom of countable choices, any two countable homogeneous $\mathbb{T}$-models in* **Set** *are isomorphic.*

# Topos-theoretic Fraïssé theorem

*Atomic topos*

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$

*Atomic site i.e.*
AP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

*Atomicity of* $\mathbb{T}'$

*Two-valued topos*

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$

JEP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

*Completeness of* $\mathbb{T}'$

*Point-of*

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$

homogeneous $\mathbb{T}$-model in **Set**

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$\mathbb{T}'$-model in **Set**

# Topological Galois theory

## Theorem

*Let $\mathbb{T}$ be a theory of presheaf type such that its category $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ of finitely presentable models satisfies AP and JEP, and let $M$ be a $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$-universal and $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$-ultrahomogeneous model of $\mathbb{T}$. Then we have an equivalence of toposes*

$$\mathbf{Sh}(f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op}, J_{at}) \simeq \mathbf{Cont}(Aut(M)),$$

*where $Aut(M)$ is endowed with the topology of pointwise convergence.*
*This equivalence is induced by the functor*

$$F : f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op} \to \mathbf{Cont}(Aut(M))$$

*sending any model $c$ of $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ to the set $\mathrm{Hom}_{\mathbb{T}\text{-}mod(\mathbf{Set})}(c, M)$ (endowed with the obvious action by $Aut(M)$) and any arrow $f : c \to d$ in $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ to the $Aut(M)$-equivariant map*

$$- \circ f : \mathrm{Hom}_{\mathbb{T}\text{-}mod(\mathbf{Set})}(d, M) \to \mathrm{Hom}_{\mathbb{T}\text{-}mod(\mathbf{Set})}(c, M) .$$

# Topological Galois theory

The following result arises from two bridges, obtained respectively by considering the invariant notions of atom and of arrow between atoms.

## Theorem

*Under the hypotheses of the last theorem, the functor $F$ is full and faithful if and only if every arrow of $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ is a strict monomorphism and it is an equivalence onto the full subcategory $\mathbf{Cont}_t(Aut(M))$ of $\mathbf{Cont}(Aut(M))$ on the transitive actions if moreover $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ is atomically complete.*

$$
\begin{tikzcd}[column sep=tiny]
 & \mathbf{Sh}(\mathrm{f.p.}\mathbb{T}\text{-}\mathrm{mod}(\mathbf{Set})^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(M)) \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\
\mathrm{f.p.}\mathbb{T}\text{-}\mathrm{mod}(\mathbf{Set})^{\mathrm{op}} & & \mathbf{Cont}_t(Aut(M))
\end{tikzcd}
$$

This theorem generalizes Grothendieck's theory of Galois categories and can be applied to obtain Galois-type theories in different fields of Mathematics, for instance one for finite groups and one for finite graphs.

# Stone-type dualities

All the classical Stone-type dualities/equivalences between special kinds of preorders and locales or topological spaces can be obtained by functorializing 'bridges' of the form

$$\begin{tikzcd} & \mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}}) \simeq \mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}}) \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ \mathscr{C} & & \mathscr{D} \end{tikzcd}$$

where $\mathscr{D}$ is a $J_{\mathscr{C}}$-dense subcategory of a preorder category $\mathscr{C}$.

For instance, take $\mathscr{D}$ equal to a Boolean algebra and $\mathscr{C}$ equal to the lattice of open sets of its Stone space for Stone duality, $\mathscr{C}$ equal to an atomic complete Boolean algebra and $\mathscr{D}$ equal to the collection of its atoms for Lindenbaum-Tarski duality.

This method also allows to generate many new dualities for other kinds of pre-ordered structures (for instance, a localic duality for meet-semilattices, a duality for $k$-frames, a duality for disjunctively distributive lattices, a duality for preframes generated by their directedly irreducible elements etc. It also naturally generalizes to the setting of arbitrary categories.

# The unification programme

The evidence provided by the results obtained so far shows that toposes can effectively act as unifying spaces for transferring information between distinct mathematical theories.

We plan to continue the research along these lines to further develop this unification programme. Central themes in this project will be:

- Deriving specific Morita-equivalences from the common mathematical practice
- Introducing new methods for generating Morita-equivalences
- Introducing new topos-theoretic invariants admitting natural characterizations
- Compiling a sort of 'encyclopedia of invariants and their characterizations' so that the 'working mathematician' can easily identify properties of theories and toposes which directly relate to his questions of interest
- Applying these methods in specific situations of interest in classical mathematics
- Automatizing the methodology 'toposes as bridges' on a computer to generate new and non-trivial mathematical results in a mechanical way

# For further reading

- A list of papers is available from my website www.oliviacaramello.com
- A book for Oxford University Press provisionally entitled *Lattices of Theories* will appear in a few months.

# International conference on topos theory


Everyone is welcome!