Isomorphism of knowledge bases: on the edge of logic and geometry

Elena Aladova

Bar-Ilan University, Israel
Penza State University, Russia

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Introduction

Classical Algebraic Geometry
Variety $Com - K$

$\Downarrow$

Universal Algebraic Geometry
Variety Θ: *Grp*, *Lie*, ...
B. Plotkin, G. Baumslag, V. Remeslennikov, ...

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Logical Geometry

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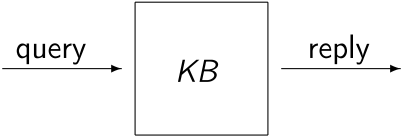
Logical Geometry

What is a knowledge base?



B.I. Plotkin: algebraically-logically-geometrical approach.

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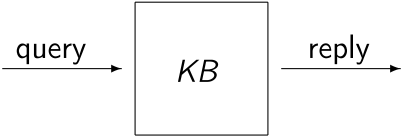

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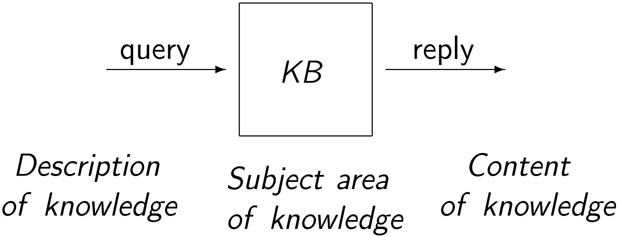

Subject area of knowledge

Subject area of knowledge is presented by a model

$$\mathcal{H} = (H, \Psi, f),$$

where

- H is an algebra in fixed variety of algebras Θ.
- Ψ is a set of relation symbols φ.
- f is an interpretation of each φ in H.

Let Θ be a variety of groups, H be a group.

- $\Psi = \{\equiv\}$.
- $\Psi = \{\equiv, <\}$.

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Description of knowledge

Description of knowledge presents a syntactical component of knowledge.

$$T = \{u(x_1, \dots, x_n)\} \subset \Phi(x_1, \dots, x_n)$$

Let H be a group with operation "$\cdot$" and $\Psi = \{\equiv\}$.

$$\neg(x_1 x_2^3 \equiv 1) \wedge (x_1 x_2 x_3^{-1} \equiv 1),$$

$$(\exists x_1 (x_1 \equiv x_2^3)) \vee (x_1 x_2 \equiv x_3).$$

If $\Psi = \{\equiv, <\}$:

$$(\exists x_1 (x_1 < x_2)) \vee (x_1 x_2 \equiv x_3).$$

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Description of knowledge
Algebra $\Phi(X)$

- $\Phi(X)$ is a boolean algebra with the operations $\vee$, $\wedge$, $\neg$.
- $\Phi(X)$ is a quantifier algebra, i.e. $\exists x_i$ is defined as unary operation, for all $x_i \in X$.
- For each $s : W(X) \to W(Y)$, there is a map $s_* : \Phi(X) \to \Phi(Y)$.

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Content of knowledge

Content of knowledge is a semantical component of a knowledge.

$$A = \{(h_1, \ldots, h_n)\} \subset H^n.$$

$$Hom(W(X), H) \;\Leftrightarrow\; H^n$$

$$\mu = (h_1, \ldots, h_n)$$

$$\Updownarrow$$

$$\mu(x_i) = h_i,\; i = 1, \ldots, n.$$

$$A \subset Hom(W(X), H).$$

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Correspondence between description and content of knowledge

Description of knowledge

$T \subset \Phi(X)$

$\Updownarrow$ via Galois correspondence

$A \subset Hom(W(X), H)$

Content of knowledge

Galois correspondence

The logical kernel of a point $\mu \in Hom(W(X), H)$ is

$$LKer(\mu) = \{u(x_1, \ldots, x_n) \in \Phi(X) \ \mid \ H \models u(\mu)\}.$$

Logical kernel of a point is an analogue of a model-theoretical type.

Galois correspondence

Algebra $\Phi(X)$		Affine space $Hom(W(X), H)$
T	$\Rightarrow$	$T^L_{\mathcal{H}}$
		$\{\mu \mid T \subset LKer(\mu)\}$
		definable set
$A^L_{\mathcal{H}}$	$\Leftarrow$	A
$\{u \mid u \in LKer(\mu)\ \forall \mu \in A\}$		
H-closed filter		

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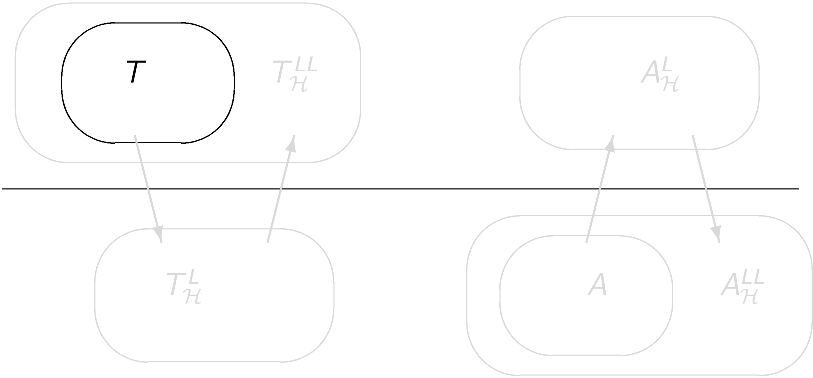
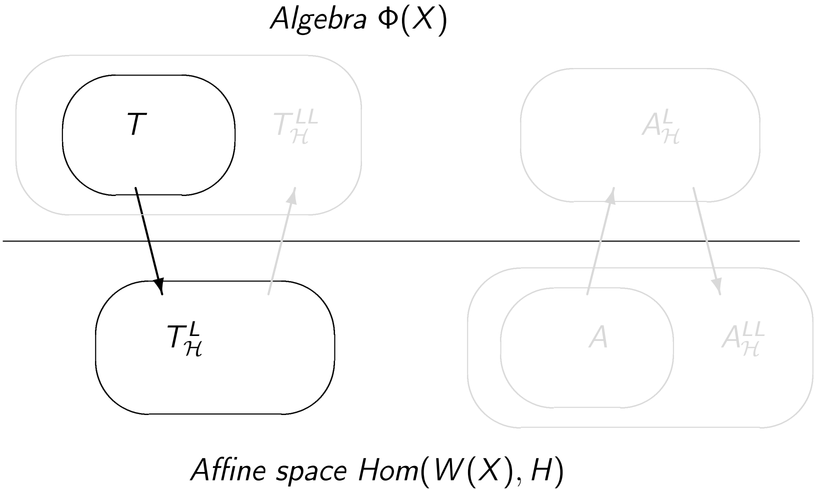


Affine space $Hom(W(X), H)$

Galois correspondence



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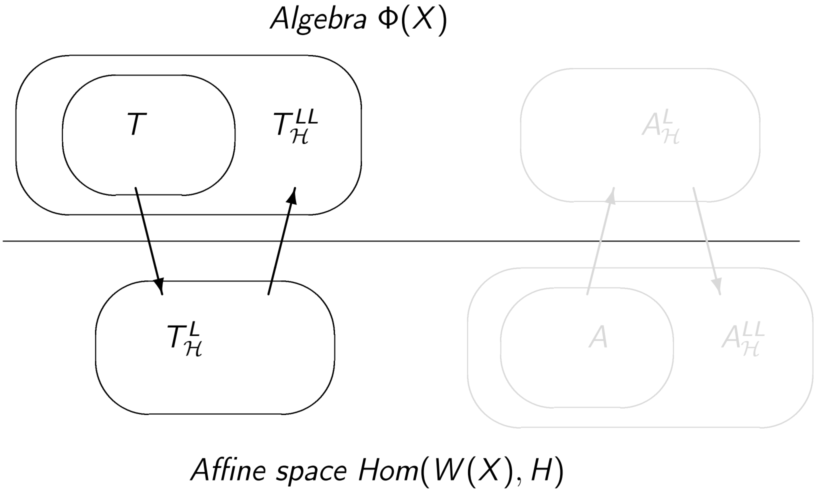

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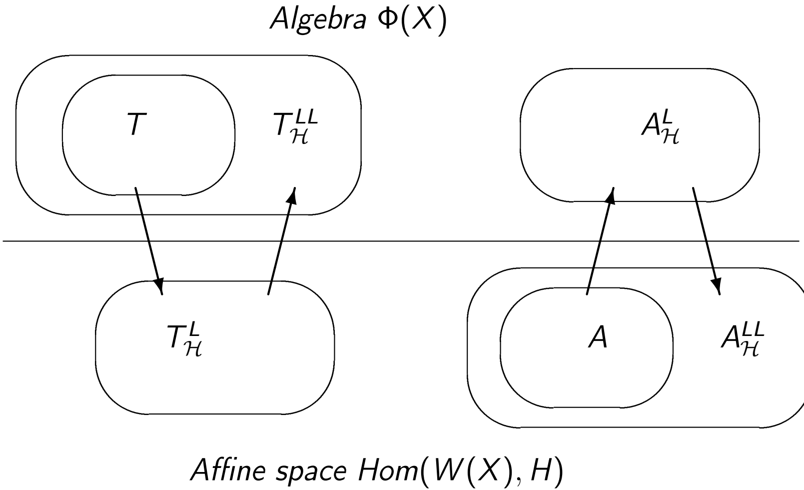

A knowledge

For a given model $\mathcal{H} = (H, \Psi, f)$, each concrete *knowledge* is a triple

$$(X, T, A),$$

where

- X is a finite set,
- T is a set of formulas from $\Phi(X)$,
- A is a set of point from $Hom(W(X), H)$ such that

$$A = T^L_{\mathcal{H}}.$$

Remark

T and $T^{LL}_{\mathcal{H}}$ describe the same content A.

A knowledge base

- *Category of knowledge descriptions* $F_\Theta(\mathcal{H})$,
- *Category of knowledge contents* $LG_\Theta(\mathcal{H})$.

Category of knowledge descriptions $F_\Theta(\mathcal{H})$

Object

$$F_\Theta^X(H)$$

is the lattice of H-closed filters in $\Phi(X)$ with fixed X.

Morphism

$$F_\Theta^Y(H) \to F_\Theta^X(H)$$

is defined using morphisms of the category of algebras $\Phi(X)$ and described above Galois correspondence.

Category of knowledge contents $LG_\Theta(\mathcal{H})$

Object

$$LG_\Theta^X(H)$$

is the lattice of all definable sets in $Hom(W(X), H)$ with fixed X.

Morphism

$$LG_\Theta^X(H) \to LG_\Theta^Y(H)$$

is defined using morphism of the category of affine spaces $Hom(W(X), H)$ and the Galois correspondence.

A knowledge base

Definition

A knowledge base $KB(\mathcal{H}) = KB(H, \Psi, f)$ *is a triple*

$$\big(F_{\Theta}(\mathcal{H}), LG_{\Theta}(\mathcal{H}), Ct_{\mathcal{H}}\big),$$

where

- $F_{\Theta}(\mathcal{H})$ *is the category of description of the knowledge,*
- $LG_{\Theta}(\mathcal{H})$ *is the category of content the knowledge,*
- $$Ct_{\mathcal{H}} : F_{\Theta}(\mathcal{H}) \to LG_{\Theta}(\mathcal{H})$$
  is a contravariant functor.

Isomorphism of knowledge bases

Problem

What are the conditions which provide an isomorphism of two knowledge bases?

Isomorphism of knowledge bases

Let $\mathcal{H}_1 = (H_1, \Psi, f_1)$ and $\mathcal{H}_2 = (H_2, \Psi, f_2)$ be given.

Definition

Knowledge bases $KB(\mathcal{H}_1)$ and $KB(\mathcal{H}_2)$ are called isomorphic if the commutative diagram takes a place

$$\begin{array}{ccc} F_\Theta(\mathcal{H}_1) & \xrightarrow{\alpha} & F_\Theta(\mathcal{H}_2) \\ {\scriptstyle Ct_{\mathcal{H}_1}}\downarrow & & \downarrow{\scriptstyle Ct_{\mathcal{H}_2}} \\ LG_\Theta(\mathcal{H}_1) & \xrightarrow{\beta} & LG_\Theta(\mathcal{H}_2) \end{array}$$

where α and β are isomorphisms of categories.

Isotypeness of knowledge bases

$$LKer(\mu) = \{u(x_1, \dots, x_n) \in \Phi(X) \mid H \models u(h_1, \dots, h_n)\}.$$

$LKer(\mu)$ is an LG-type of μ.

Let $S^X(\mathcal{H})$ be the set of all X-LG-types of a model $\mathcal{H}$.

Definition

Models $\mathcal{H}_1 = (H_1, \Psi, f_1)$ and $\mathcal{H}_2 = (H_2, \Psi, f_2)$ are called LG-isotypic if

$$S^X(\mathcal{H}_1) = S^X(\mathcal{H}_2),$$

for each finite $X \in \Gamma$.

Isotypeness of knowledge bases

Let two knowledge bases $KB(H_1, \Psi, f_1)$ and $KB(H_2, \Psi, f_2)$ be given.

Theorem

If models (H_1, Ψ, f_1) and (H_2, Ψ, f_2) are LG-isotypic then the corresponding knowledge bases $KB(H_1, \Psi, f_1)$ and $KB(H_2, \Psi, f_2)$ are isomorphic.