States on finite GBL*algebras

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Introduction

Classically, a probability measure is defined on a sigma-algebra of events, that is a countably complete Boolean algebra.

As there is a correspondence between events and sentences expressing them, one may wonder how probability can be associated to events expressed by a non-classical propositional logic.

In fact, analog notions have been given, for instance the notion of *state* on a MV-algebra [Mundici '95], finitely additive probability measure on Gödel algebras [Aguzzoli, Gerla, Marra '08] and more generally the notion of *valuation* [Rota '64, '73].

GBL-algebras are the divisible residuated lattices, and their variety extends the ones of Heyting algebras and MV-algebras.

In our ongoing work [3, 4] we give a representation of finite GBL-algebras in terms of *GBL-pairs*, which are Heyting algebras with certain equivalence relations induced by groups of automorphisms.

We used the representation of finite GBL-algebras as labelled posets [Jipsen, Montagna '09] and similar models of algebraic structures with equivalence relations [Jenča '07][Vetterlein '08].

In [Di Nola, Holčapek, Jenča '09] a representation of this kind is extended to a functorial correspondence.

Aiming to generalize it to GBL-algebras, we found more natural to make explicit one further operation, thus defining *GBL*algebras*.

The representation of GBL*algebras as GBL-pairs allows us to define a notion of state on GBL*algebras which generalizes the existing ones, and it suggests an interpretation of state based on Kripke models for intuitionistic logic.

We make use of this representation to investigate some properties of a state, in particular the relation between the state and the density function, and the expression of the state in terms of *conditional states*

Outline

- Posets and Heyting algebras
- States on Heyting algebras
- Interpretation of states on Heyting algebras
- GBL*algebras
- GBL-pair representation of GBL*algebras
- States on GBL*algebras
- Interpretation of states on GBL*algebras
- Conditional states

Posets and Heyting algebras

Let $\langle P, \leq \rangle$ be a poset.

For $x \in P$, let $\downarrow x = \{y \in P \mid y \leq x\}$.
$X \subseteq P$ is a downset if $\downarrow x \subseteq X$ for all $x \in X$.
Let $D(P)$ be the set of downsets of P.
Let $\top = P$, $\bot$ be the empty subset and

$$A \to B = \bigvee \{X \in D(P) \mid (X \wedge A) \leq B\}$$

where $\wedge$ and $\vee$ are intersection and union of sets.

Let $\langle H, \wedge, \vee, \to, \top, \bot \rangle$ be a Heyting algebra.
An element $x \in H$, $x \neq \bot$, is join-prime if,
for all $a, b \in H$, $x \leq a \vee b$ implies $x \leq a$ or $x \leq b$.
Let $J(H)$ be the set of join-prime elements of H.

Theorem. (reformulation of Birkhoff representation)

For every poset P,
$\langle D(P), \wedge, \vee, \rightarrow, \top, \bot \rangle$ is a Heyting algebra and
$\langle J(D(P)), \leq \rangle \cong \langle P, \leq \rangle$.
Conversely,
for every finite Heyting algebra H,
$\langle H, \wedge, \vee, \rightarrow, \top, \bot \rangle \cong \langle D(J(H)), \wedge, \vee, \rightarrow, \top, \bot \rangle$.

$J(f)$: restriction of f to $J(H)$,
for f automorphism of H.
$J(f)$ is an order automorphism.

Every order automorphism g of $J(H)$ can be extended in a unique way to an automorphism of H.

States on Heyting algebras

We define *state* as a weak notion of *probability*.

Definition.

Let H be a finite Heyting algebra.
A state on H is a function $v : H \to [0, 1]$ such that

$v(\bot) = 0$
$v(\top) = 1$
if $a \leq b$, then $v(a) \leq v(b)$
$v(a \wedge b) + v(a \vee b) = v(a) + v(b)$
for all $a, b \in H$.

A density on a finite poset P is a function $d : P \to [0,1]$ such that $\sum_{x \in P} d(x) = 1$.

States on a Heyting algebra H and densities on $J(H)$ are related by the Möbius inversion formula (see [8]). If $d : J(H) \to [0,1]$ is a density, then $v : H \to [0,1]$ is a state, where

$$v(x) = \sum_{i \in J(H), i \leq x} d(i);$$

if $v : H \to [0,1]$ is a state, then $d : J(H) \to [0,1]$ is a density, where

$$d(x) = \sum_{i \in J(H), i \leq x} \mu(i,x) v(i);$$

and μ is the Möbius function (see [8]).
The two transformations are inverse to each other.

Interpretation

Classically, probability is defined on a Boolean algebra of events. If events are *denoted* by sentences of a propositional logic, it is natural to consider also other relational structures for events.

Here we give the following interpretation:

An element $x \in P$ represents a *possible case*;
The relation $x \leq y$ represents the fact that all the events realized in y are realized in x as well;
an event E is interpreted as the set of all the cases in which E is realized, that is a downset of P.

The probabilities of the possible cases are not subject to any logical constraint, and the density function can be interpreted as a prior probability distribution.

Finite GBL*algebras

A finite GBL*algebra is a structure
$\langle X, \odot, \oplus, \to, \top, \bot \rangle$
such that X is a finite set and
$\langle X, \odot, \top \rangle$ is a commutative monoid
$\langle X, \oplus, \bot \rangle$ is a commutative monoid
$a \to (a \oplus b) = \top$
$a \odot (a \to b) = b \odot (b \to a)$
$(a \odot b) \to c = b \to (a \to c)$
$(a \oplus b) \to b = a \to (a \odot b)$.

Derived operations are
$a \wedge b = a \odot (a \to b) = b \odot (b \to a)$
$a \vee b = (a \oplus b) \wedge (a \to b) \to b = (a \oplus b) \wedge (b \to a) \to a$.
With these operations,
$\langle X, \odot, \vee, \to, \top, \bot \rangle$ is a GBL-algebra (see [4]).

Let X be a finite GBL*algebra.

An element $a \in X$ is idempotent if $a \odot a = a \oplus a$.
We call skeleton of X the set of idempotents:

$$I(X) = \{a \in X \mid a \odot a = a \oplus a\}$$

$I(X)$ is closed with respect to $\odot, \oplus, \to, \top, \bot$;
$a \odot b = a \wedge b$ and $a \oplus b = a \vee b$ for all $a \in I(X)$, $b \in X$;
$\langle I(X), \wedge, \vee, \to, \top, \bot \rangle$ is a Heyting algebra.

If $I(X) = X$, then X is a Heyting algebra.
Conversely, every Heyting algebra $\langle H, \wedge, \vee, \to, \top, \bot \rangle$ is a GBL*algebra, by setting $a \odot b = a \wedge b$ and $a \oplus b = a \vee b$.

If $I(X)$ is a Boolean algebra, then X is a MV-algebra, and all finite MV-algebra are such kind of GBL*algebras.

GBL-pairs

Let H be a finite Heyting algebra,

G be a subgroup of $Aut(H)$,
$\sim \subseteq H \times H$ be the equivalence relation:
$a \sim b$ if there is $g \in G$ such that $g(a) = b$,
$[a]$ be the class of a,
H/G be the quotient set with the order relation:
$[a] \leq [b]$ if there is $c \sim a$ such that $c \leq b$,

$J(G)$ be the group of $\{J(g) \mid g \in G\}$, isomorphic to G,
$\sim \subseteq J(H) \times J(H)$ be the equivalence relation:
$x \sim y$ if there is $g \in G$ such that $J(g)(x) = y$,
$[x]$ be the class of x,
$J(H)/J(G)$ be the quotient set with the order relation:
$[x] \leq [y]$ if there is $z \sim x$ such that $z \leq y$.

G is chain-transitive on H if, for every pair of chains

$$a_1 < a_2 < \ldots < a_k \in H, \qquad b_1 < b_2 < \ldots < b_k \in H$$

such that $a_i \sim b_i$ for all i, $1 \leq i \leq k$,
there is $g \in G$ such that $g(a_i) = b_i$ for all i, $1 \leq i \leq k$.

In other words, if two chains are elementwise equivalent, they are equivalent through the same automorphism.

Definition.

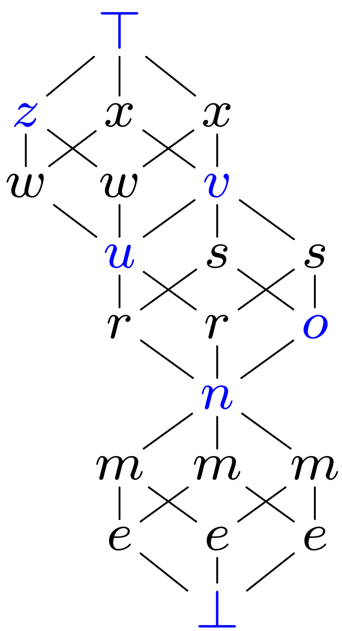
A finite GBL-pair is a pair (H, G) such that
H is a finite Heyting algebra and
G is a subgroup of $Aut(G)$ chain-transitive on H.

This definition generalizes, for the finite case, the definition of MV-pairs in [5], where Boolean algebras instead of Heyting algebras are considered.

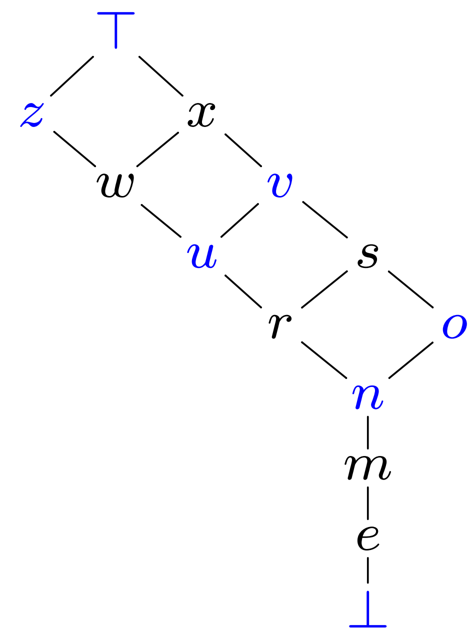
Example 1.

The following are a Heyting algebra H with an equivalence relation given by a group G (equivalent elements are labeled by the same letter) and the quotient H/G.

H :



H/G :



A more direct characterization of GBL-pairs can be given on the restriction to join-prime elements.

Let (H, G) be a finite GBL-pair.

Proposition 1.

$J(G)$ is the group of bijective functions $f : J(H) \to J(H)$ such that $f([x]) = [x]$ for all $[x] \in J(H)/J(G)$.

The proof relies on a suitable totally ordered extension of $J(H)$, then reasoning by induction on the total order.

Corollary.

For all $x, y \in J(H)$,
$x < y$ if, and only if, $[x] < [y]$.

On the other hand, let

H be a finite Heyting algebra;
$\sim \subseteq J(H) \times J(H)$ be an equivalence relation such that, for all $x, y \in J(H)$, $x < y$ if, and only if, $[x] < [y]$;
F be a group of bijective functions $f : J(H) \to J(H)$ such that $f([x]) = [x]$ for all $[x] \in J(H)/J(G)$.

Proposition 2.

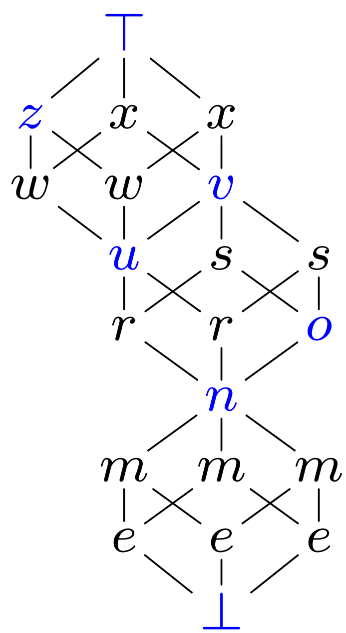
$F = J(G)$, where G is a subgroup of $Aut(H)$;
(H, G) is a GBL-pair.

Proposition 1 and *Proposition 2* imply that G is univocally determined by the equivalence relation $\sim$. Hence, we can identify (H, G) and H/G.

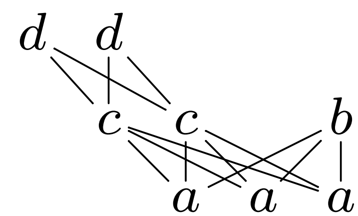
Example.

Let H be as in *Example 1*.
The following is $J(H)$, with equivalent elements labelled by the same letter.

H :



$J(H)$:



We can recover the GBL*algebra operations between two classes in H/G by means of infima and suprema of classes of the results of Heyting operations performed between elements in the two classes.

For $[a], [b] \in H/G$ let
$[a] \odot [b] = \bigwedge \{[x \wedge y] \mid x \sim a,\ y \sim b\}$
$[a] \oplus [b] = \bigvee \{[x \vee y] \mid x \sim a,\ y \sim b\}$
$[a] \rightarrow [b] = \bigvee \{[x \rightarrow y] \mid x \sim a,\ y \sim b\}$

Proposition.

$\langle H/G, \odot, \oplus, \rightarrow, [\top], [\bot]\rangle$ is a GBL*algebra.

Derived connectives turn out to be:
$[a] \wedge [b] = \bigvee \{[x \wedge y] \mid x \sim a,\ y \sim b\}$
$[a] \vee [b] = \bigwedge \{[x \vee y] \mid x \sim a,\ y \sim b\}$

The following is a representation theorem for finite GBL*algebras.

Theorem.

Let let X be a finite GBL*algebra.
Then, there is a GBL-pair (H, G) such that
$\langle X, \odot, \oplus, \to, \top, \bot\rangle \cong \langle H/G, \odot, \oplus, \to, [\top], [\bot]\rangle$.

The proof relies on a representation of GBL*algebras as labelled (or weighted) posets (see [3], [6]).

By Birkhoff duality applied to H, the representation can be given by equivalence classes of downsets of a poset isomorphic to $J(H)$.

States on GBL*algebras

We generalize the notion of *state* from Heyting to GBL*algebras.

Definition.

Let X be a finite GBL*algebra.
A state on H is a function $s : H \to [0, 1]$ such that

$s(\bot) = 0$
$s(\top) = 1$
if $a \leq b$, then $s(a) \leq s(b)$
$s(a \odot b) + s(a \oplus b) = s(a) + s(b)$
for all $a, b \in H$.

Properties:

$s(a \wedge b) + s(a \vee b) = s(a) + s(b)$;
s restricted to $I(X)$ is a state of Heyting algebra;
if X is a MV-algebra, then s is a state of MV-algebra, as defined in [7];
if X is a Gödel algebra, then s is a finitely additive probability in the sense of [1].

Proposition.

Let X be a finite GBL*algebra and $v : J(H) \to [0, 1]$ a state on its Heyting skeleton. Then, v can be extended uniquely to a state $s : H \to [0, 1]$.

The proof makes use of a chain decomposition of X (see also [6])

Definition.

Let (H, G) be a finite GBL-pair
and $v : H \to [0, 1]$ a state on H.
v is invariant with respect to G if
$a \sim b$ implies $v(a) = v(b)$ for all $a, b \in H$.

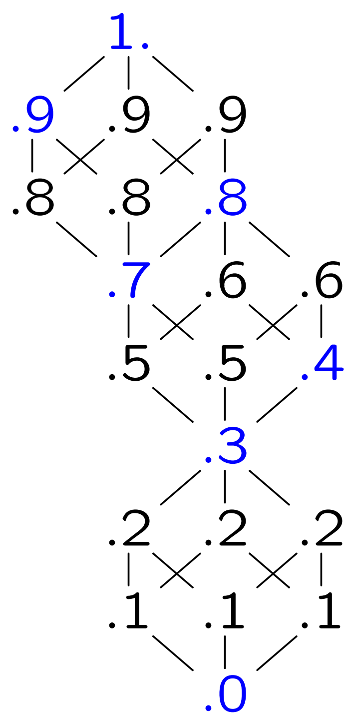
Proposition.

Let (H, G) be a GBL-pair and
v an invariant state on H.
Then, the function $s : H/G \to [0, 1]$, $s([x]) = v(x)$
is a state on the GBL*algebra H/G.
Conversely, let $s : H/G \to [0, 1]$ be a state on H/G.
Then, the function $v : H \to [0, 1]$, $v(x) = s([x])$
is a G-invariant state on H.

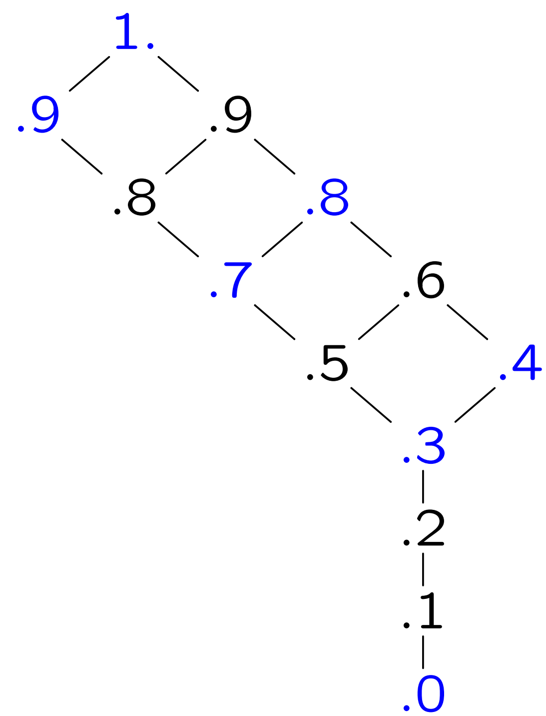
Example.

Let H and H/G be as in *Example 1*.
The following are an invariant state on H and the corresponding state on H/G.

v :



s :



In a similar way we say a density $d : J(H) \to [0,1]$ is invariant with respect to $J(G)$ when $a \sim b$ implies $d(a) = d(b)$ for all $a, b \in J(H)$. Invariance on H and on $J(H)$ are related by Möbius inversion formula.

Proposition.

If $d : J(H) \to [0,1]$ is an invariant density,
then the state $v : H \to [0,1]$ is invariant, where

$$v(x) = \sum_{i \in J(H), i \leq x} d(i).$$

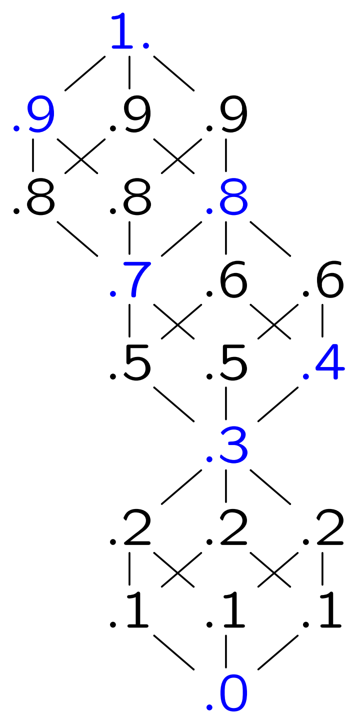
Conversely, if $v : H \to [0,1]$ is an invariant state,
then the density $d : J(H) \to [0,1]$ is invariant, where

$$d(x) = \sum_{i \in J(H), i \leq x} \mu(i,x) v(i).$$

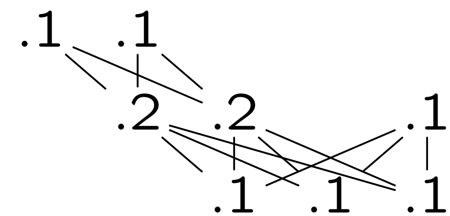
Example.

Let H and $J(H)$ be as in *Example 2*.
The following are an invariant state on H and the corresponding invariant density on $J(H)$.

v :



d :



Interpretation

Starting from the relational model for a Heyting algebra of events previously described, we make the assumption that sentences denoting events are not *sharp*, hence some events are *indiscernible*, and they are partitioned into equivalence classes of indiscernibility.

The equivalence between two cases involves also their relations with all the other cases, hence it is determined by an automorphism. Chain-transitivity is a further strengthening of the relation of indiscernibility.

If a prior probability is assigned, it is natural to think that indiscernible events have the same probability and indiscernible events have the same density.

Conditional states

Let P be a finite poset. We define a function $e : P \to \mathbb{Z}$ as the opposite of the Möbius function on the opposite of P, closed with a least point. Namely,

$$e(x) = 1 - \sum_{y>x} e(y)$$

Proposition 3.

Let X be a finite GBL*algebra and $s : X \to [0, 1]$ be a state on X. Then,

$$s(x) = \sum_{j \in J(I(X))} e(j)s(x \wedge j).$$

It follows from Möbius inversion formula and the representation of a GBL*algebra as a GBL-pair.

Let X be a finite GBL*algebra and $s : X \to [0,1]$ be a strictly monotone state on X, that is $a < b$ implies $s(a) < s(b)$ for all $a, b \in X$.

s corresponds to a strictly positive density on $J(H)$, where (H, G) is the GBL-pair representing X.

For every $x \in X$ and $i \in I(X)$ it is well defined:

$$s(x \mid i) = \frac{s(x \wedge i)}{s(i)}.$$

We call the function $s(\cdot \mid i)$ a conditional state with respect to the idempotent i.

If X is a MV-algebra, $s(\cdot \mid i)$ coincides with an established notion of conditional state (see [7]).

For a strictly monotone state $s : X \to [0,1]$, the formula in *Proposition 3* becomes:

$$s(x) = \sum_{j \in J(I(X))} e(j) s(x|j) s(j),$$

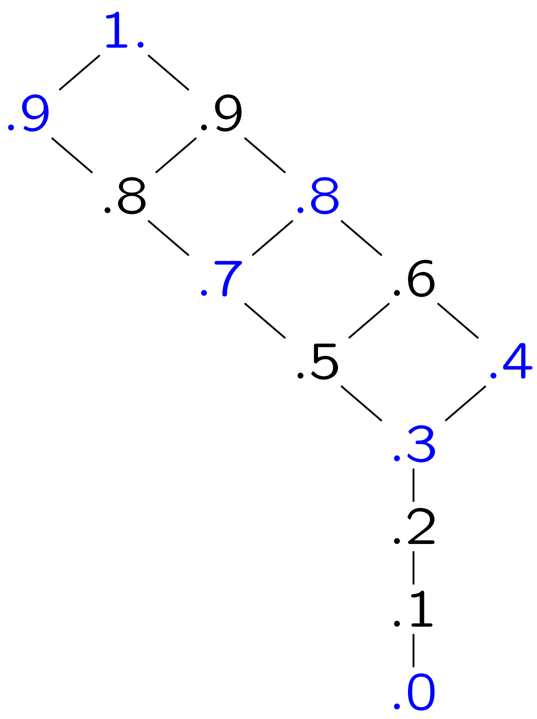
which recalls the law of total probability.
If X is a MV-algebra, then $e(j) = 1$ for all j, hence

$$s(x) = \sum_{j \in J(I(X))} s(x|j) s(j).$$

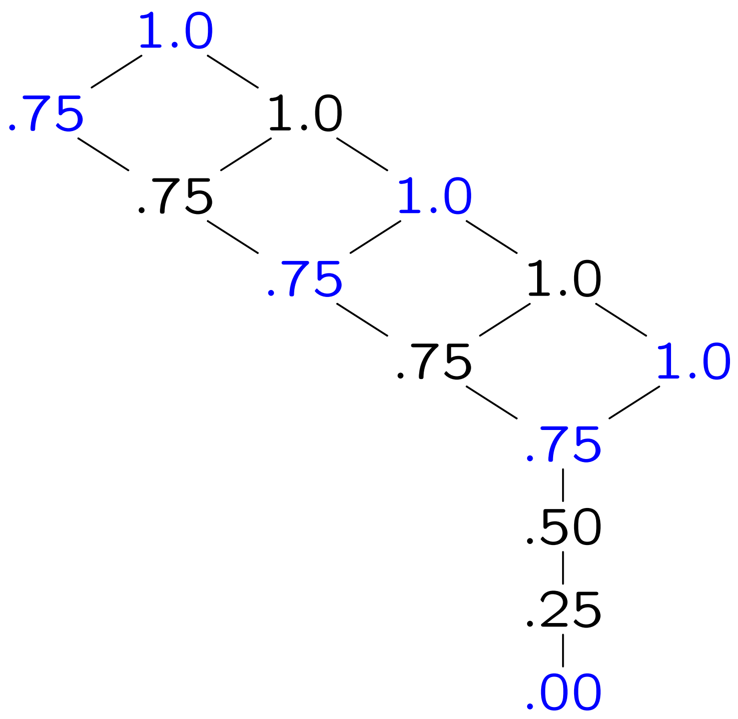
In this case $s(x|j)$ corresponds to a truth valuation, that is a homomorphism to the generic algebra $[0,1]$, and all truth valuation arise in this way as j varies in $J(I(X))$. This suggests to think of the conditional state as a truth valuation for any GBL*algebra, although the connection with homomorphisms is lost.

Example.

$s(\cdot)$



$s(\cdot \mid j)$



Conclusion

We have defined a notion of state, or finitely additive probability measure, which generalizes analog established notions on classes of algebras such as Boolean algebras, Gödel algebras and MV-algebras. Indeed we defined states on finite GBL*algebras, which have the previous as particular finite cases.

We investigated the properties of states on GBL*algebras through their representation as GBL-pairs and Birkhoff's representation. To do so, we applied combinatorial methods such as the Möbius inversion formula, that could not be easily generalizable to the infinite case.

Representation theorems allow us to think of the state in terms of distribution, being also a support for interpretation.

We made few assumptions on the distribution. We didn't make full use of the logical structure of algebras, in particular the implication.

On this point other approaches exist, but we believe that also the one presented here can be adapted to be logically more expressive.

THANK YOU

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