Endomorphism monoids of ω-categorical structures

Michael Kompatscher

michaelkompatscher@hotmail.com

Institut of Computer Languages
Technische Universität Wien

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Countable, ω-cat. structures $\mathcal{A}$ and $\mathcal{B}$ are interdefinable iff

$$\mathrm{Aut}(\mathcal{A}) = \mathrm{Aut}(\mathcal{B})$$

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- Can we reconstruct the topology of $\mathrm{Aut}(\mathcal{A})$?

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Reconstruction

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- transformation monoid
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No!

(Evans + Hewitt '90; Bodirsky + Evans + Pinsker + MK '15)

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Profinite groups are closed permutation groups where every orbits contains finitely many elements.

Example (Witt '54)

There are two separable profinite groups G, G' that are isomorphic, but not topologically isomorphic.

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Lemma (Hrushovski)

There is a oligomorphic Φ such that for every separable profinite group R there is an oligomorphic Σ_R:

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Proof idea: $R \leq \prod_{n\geq 1} \mathrm{Sym}(n)$.

Look at finite sets. Partition the n-tuples into partition classes $P_1^n, P_2^n, \ldots P_n^n$ for all $n \geq 1$. This gives us a Fraïssé-class.

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We can think of Σ acting on the partition classes $P_1^n, P_2^n, \ldots P_n^n$.

This gives us $\Sigma / \Phi \cong^T \prod_{n \in \mathbb{N}} \mathrm{Sym}(n)$. □

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The real proof deviates from the above.

Lifting to the monoid closure

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The quotient homomorphism $\Sigma_R \to R$ extends to a continuous monoid homomorphism

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We get:

Result for monoids

$\overline{\Sigma_G}$ and $\overline{\Sigma_{G'}}$ are isomorphic, but not topologically isomorphic.

Oligomorphic clones

Observation

Let $I : \Gamma \to \Delta$ be a monoid homomorphism. If I sends constants to constants, it has a natural extension to a clone homomorphism $\mathrm{Clo}(\Gamma) \to \mathrm{Clo}(\Delta)$.

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The clones $\mathrm{Clo}(\overline{\Sigma_G})$ and $\mathrm{Clo}(\overline{\Sigma_{G'}})$ are isomorphic but not topologically isomorphic.

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Let $l : \Gamma \to \Delta$ be a monoid homomorphism. If l sends constants to constants, it has a natural extension to a clone homomorphism $\text{Clo}(\Gamma) \to \text{Clo}(\Delta)$.

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The clones $\text{Clo}(\overline{\Sigma_G})$ and $\text{Clo}(\overline{\Sigma_{G'}})$ are isomorphic but not topologically isomorphic.

This answers a question by Bodirsky, Pinsker and Pongrácz.

Thank you!