Dual characterizations for finite lattices via correspondence theory for monotone modal logic

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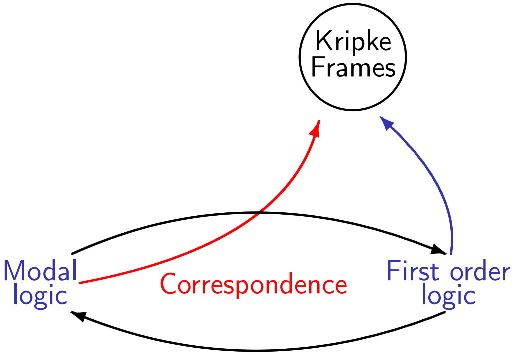
May 1, 2014

TACL 2015, Ischia, Italy

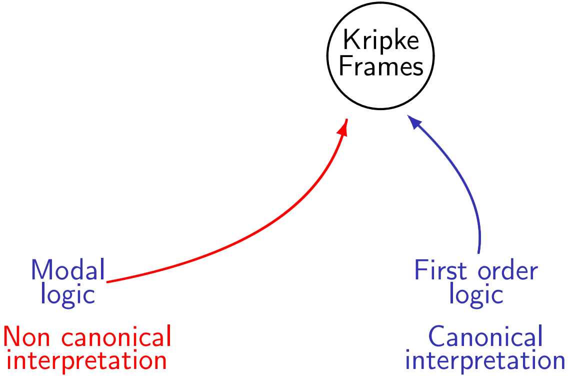
Correspondence via Duality

Correspondence theory arises semantically:



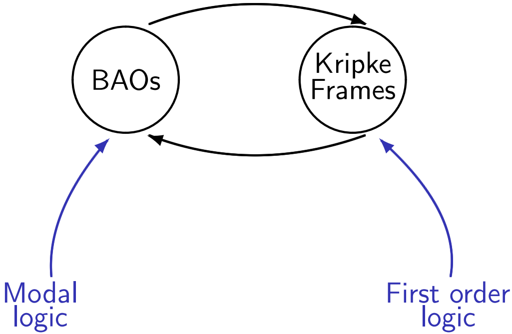
Correspondence via Duality

An asymmetry:



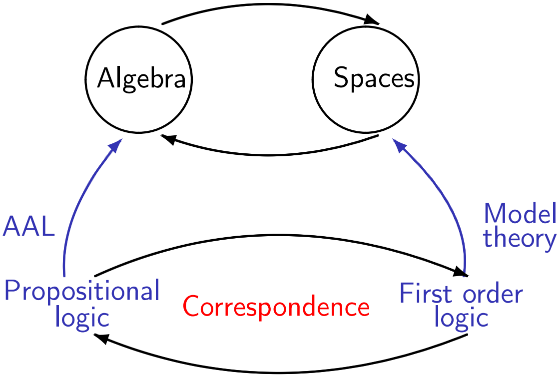
Correspondence via Duality

Symmetry re-established via duality:



Correspondence via Duality

Correspondence available <u>not</u> just for normal modal logic:



Unified correspondence

Hybrid logics
[CR15]

DLE-logics
[CP12, CPS]

Substructural logics
[CP15]

Mu-calculi
[CFPS15, CGP14, CC15]

Display calculi
[GMPTZ]

Regular DLE-logics
Kripke frames with
impossible worlds
[PSZ15a]

Jónsson-style vs
Sambin-style canonicity
[PSZ15b]

Finite lattices and
monotone ML
[FPS15]

Canonicity via
pseudo-correspondence
[CPSZ]

Aim: Algorithmic Correspondence Theory for Finite Lattices

Theorem [Nation 90]

$t_n \leq s_n$ sequence of lattice inequalities generalizing distributivity, L a lattice, and $D \subseteq J(L) \times J(L)$ binary relation (defined in Nation 90).

$$L \models s_n \leq t_n \quad \text{iff} \quad \text{there is no } D\text{-chain of length } n \text{ in } L$$

Aim: Embed Nation's theorem into correspondence theory for lattices

Strategy:

1. Finite lattices as relational structures (join-presentations)
2. Join-presentations as enriched monotone neighbourhood frames
3. Develop ALBA for monotone neighbourhood frames
4. Develop ALBA for enriched monotone neighbourhood frames

Finite lattices as relational frames

L finite lattice, $J(L)$ set of join-irreducible elements of L.
Def: $j \in J(L)$, $C \subseteq J(L)$.

$$C \text{ is a cover of } j \quad \text{iff} \quad j \leq \bigvee C$$

Def: The join-presentation of L is

$$\langle J(L), \leq, \sigma : J(L) \longrightarrow \mathcal{P}\mathcal{P}J(L)\rangle$$

s.t. for every $j \in J(L)$,

$$\sigma(j) := \{C \subseteq J(L) \mid j \leq \bigvee C\}.$$

Def: $S \subseteq J(L)$ is closed if $\forall j \in J(L)$ ($S \in \sigma(j) \Rightarrow j \in S$).

Theorem [Santocanale2009]

Every finite lattice L is isomorphic to the lattice of its closed sets.

Join-presentations as enriched monotone neighbourhood frames

Def:

- Kripke frame: $\langle W, R : W \longrightarrow \mathcal{P}W \rangle$
- Neighbourhood frame: $\langle W, N : W \longrightarrow \mathcal{P}\mathcal{P}W \rangle$
- Monotone neighbourhood frame: $\langle W, N : W \longrightarrow \mathcal{P}\mathcal{P}W \rangle$ such that, for every $w \in W$, $N(w)$ is upward closed.
  i.e. $S \subseteq T$ and $S \in N(w)$ implies $T \in N(w)$.

join-presentation of L: $\langle J(L), \leq, \sigma : J(L) \longrightarrow \mathcal{P}\mathcal{P}J(L) \rangle$ with $\sigma(j) := \{C \subseteq J(L) \mid j \leq \bigvee C\}$.

$\langle J(L), \sigma : J(L) \longrightarrow \mathcal{P}\mathcal{P}J(L) \rangle$ is a monotone neighbourhood frame.

Join-presentations as enriched monotone neighbourhood frames

Def:

- Monotone neighbourhood frame: $\langle W, N : W \longrightarrow \mathcal{P}\mathcal{P}W \rangle$ such that, for every $w \in W$, $N(w)$ is upward closed.
- Enriched monotone neighbourhood frame:

$$\langle W, \leq, N : W \longrightarrow \mathcal{P}\mathcal{P}W \rangle$$

s.t.

- $(W, \leq)$ is a poset
- $\langle W, N : W \longrightarrow \mathcal{P}\mathcal{P}W \rangle$ is a monotone neighbourhood frame

$\langle J(L), \ \leq, \ \sigma : J(L) \longrightarrow \mathcal{P}\mathcal{P}J(L) \rangle$ is an enriched monotone neighbourhood frame.

Duality-induced translation: lattice terms $\rightarrow$ modal fm'as

Monotone neighbourhood frames $\langle W, N : W \longrightarrow \mathcal{P}\mathcal{P}W \rangle$ are natural models for the modal language:

$$\varphi ::= \bot \mid \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid (\exists\forall)\varphi \mid (\forall\exists)\varphi.$$

$$\mathbb{M}, w \Vdash (\exists\forall)\varphi \quad \text{iff} \quad \exists C \in N(w),\ \forall c \in C,\ \mathbb{M}, c \Vdash \varphi$$
$$\mathbb{M}, w \Vdash (\forall\exists)\varphi \quad \text{iff} \quad \forall C \in N(w),\ \exists c \in C,\ \mathbb{M}, c \Vdash \varphi.$$

Lattice terms $\longrightarrow$ modal formulas

Language of lattices: $\varphi ::= \bot \mid \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi$

$$ST(p) = p, ST(\top) = \top, ST(\bot) = \bot, ST(t \wedge s) = ST(t) \wedge ST(s),$$

$$ST(t \vee s) = (\exists\forall)\ (ST(t) \vee ST(s)).$$

ALBA for monotone modal logic

ALBA: algorithm based on

- adjunction, residuation, minimal/maximal valuation argument
- Ackerman Lemma (to eliminate propositional variables)

Monotone modal logic: $(\exists\forall)$ and $(\forall\exists)$ are non-normal modalities $\longrightarrow$ no adjunction $\longrightarrow$ no ALBA

Solution [Hansen 2003]:
$(\exists\forall)X \sim \Diamond_N \Box_{\ni} X \sim \{w \in W \mid \exists C \in N(w),\ \forall w' \in C,\ w' \in X\}$

Two-sorted frames: $\mathbb{X} = \langle X, Y, R_{XY}, R_{YX} \rangle$

ex: $\langle\ W,\ \mathcal{P}W,\ R_N\ \subseteq\ W \times \mathcal{P}W,\ \ni\ \subseteq\ \mathcal{P}W \times W\ \rangle$

ex: $\langle\ J(L),\ \mathcal{P}J(L),\ R_\sigma\ \subseteq\ J(L) \times \mathcal{P}J(L),\ \ni\ \subseteq\ \mathcal{P}J(L) \times J(L)\ \rangle$

ALBA for monotone modal logic

two-sorted frame:

$\langle\, J(L),\ \mathcal{P}J(L),\ R_\sigma\ \subseteq\ J(L) \times \mathcal{P}J(L),\ \ni\, \subseteq\ \mathcal{P}J(L) \times J(L)\,\rangle$

Formulas: $\varphi ::= \bot \mid \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond_\sigma \Box_\ni \varphi \mid \Box_\sigma \Diamond_\ni \varphi.$

Syntax in ALBA

$\varphi ::= \bot \mid \top \mid p \mid \mathbf{j} \mid \mathbf{m} \mid \underline{\mathbf{j}} \mid \underline{\mathbf{m}} \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \smallsetminus \varphi \mid \varphi \to \varphi \mid \Diamond_\sigma \varphi \mid \Box_\sigma \varphi \mid \Diamond_\ni \varphi \mid \Box_\ni \varphi \mid \blacklozenge_\sigma \varphi \mid \blacksquare_\sigma \varphi \mid \blacklozenge_\ni \varphi \mid \blacksquare_\ni \varphi.$

ALBA for join-presentations: we add some rules describing the specific behaviour of join-presentations.

ALBA for monotone modal logic

SOME RULES

$$\frac{\varphi \leq \psi_1 \wedge \psi_2}{\varphi \leq \psi_1 \;\;\&\;\; \varphi \leq \psi_2} \qquad \frac{\alpha \wedge \beta \leq \gamma}{\alpha \leq \beta \to \gamma} \qquad \frac{\alpha \leq_Y \Box_\ni \beta}{\blacklozenge_\ni \alpha \leq_X \beta}$$

Ackermann Rule:

$$\frac{\exists p \left[\&_{i=1}^{n} \{ p \leq \alpha_i \} \;\;\&\;\; \&_{j=1}^{m} \{ \beta_j(p) \leq \gamma_j(p) \} \right]}{\&_{j=1}^{m} \{ \beta_j((\bigwedge_{i=1}^{n} \alpha_i)/p) \leq \gamma_j((\bigwedge_{i=1}^{n} \alpha_i)/p) \}} (LAR)$$

where p does not occur in $\alpha_1, \ldots, \alpha_n$, the formulas $\beta_1(p), \ldots, \beta_m(p)$ are negative in p, and $\gamma_1(p), \ldots, \gamma_m(p)$ are positive in p.

ALBA for join-presentations

Ackermann Rule:

$$\frac{\exists p \left[\&_{i=1}^{n}\{\alpha_i \leq p\} \;\; \& \;\; \&_{j=1}^{m}\{\beta_j(p) \leq \gamma_j(p)\}\right]}{\&_{j=1}^{m}\{\beta_j((\langle\lhd\rangle[\ni]\langle\leq_X\rangle \bigvee_{i=1}^{n}\alpha_i)/p) \leq \gamma_j((\langle\lhd\rangle[\ni]\langle\leq_X\rangle \bigvee_{i=1}^{n}\alpha_i)/p)\}}$$

Lemma:

$$(S1) := \left(\begin{array}{l} \mathbf{j} \leq \langle\lhd\rangle\mathbf{C} \\ \mathbf{k} \leq \langle\in\rangle\mathbf{C} \\ \langle\leq_J\rangle\mathbf{j} \wedge \langle\leq_J\rangle\mathbf{k} \leq \kappa(\mathbf{k}) \\ \langle\leq_J\rangle\mathbf{k} \wedge s \leq \kappa(\mathbf{k}) \end{array}\right),$$

$$(S2) := \left(\begin{array}{l} (S1) \\ \mathbf{j} \wedge \langle\lhd\rangle[\ni](\langle\lhd\rangle[\ni]\langle\leq_J\rangle((\langle\in\rangle\mathbf{C}\setminus\mathbf{k}) \vee (\langle\leq_J\rangle\mathbf{j} \wedge \langle\leq_J\rangle\mathbf{k}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \vee(\langle\leq_J\rangle\mathbf{k} \wedge s)) \leq \bot \end{array}\right).$$

Result & Open questions

Results:

- ALBA for monotone modal logic
- Obtain a new correspondence result similar to Nation's result
- New Ackermann type rule based on a minimal/maximal valuation argument

Open questions:

- Adapt ALBA for non-normal modal logics
- Generalize the new Ackermann type rule

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