SGF-quantales and their groupoids

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joint work with Helle Hansen, Alessandra Palmigiano and Riccardo Re

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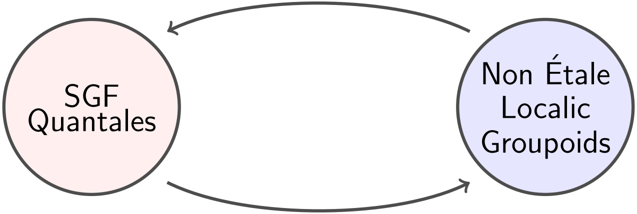
Work in progress

Bijective correspondence:



Quantales and Groupoids

	Étale	Non Étale
Point-set		Spatial SGF-Quantales & "Set" Groupoids + Bases [Palmigiano & Re 2011]
Point-free	Inverse Quantal Frames & Localic Étale Groupoids [Resende 2007]	SGF-Quantales & Localic Groupoids

Quantales

Definition

Quantale= sup lattice + associative product $(a, b) \mapsto ab$ satisfying

$$a(\vee b_i) = \vee(ab_i)$$

$$(\vee a_i)b = \vee(a_i b)$$

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Definition

Q is unital if $\exists e \in Q$ s.t.

$$qe = q = eq$$

Q is involutive if $\exists (-)^{\dagger} \colon Q \to Q$ sup lattice map st

$$x^{\dagger\dagger} = x$$

$$(xy)^{\dagger} = y^{\dagger} x^{\dagger}$$

Example

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Subrelations of eq rel $R \subset X \times X$

- Join $=$ union
- Product: $xSTy$ iff $\exists z$ st xSz and zTy
- $e = \Delta$ diagonal relation
- $S^{\dagger} = \{(y, x) \mid xSy\}$.

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Definition

A homomorphism of (involutive) quantales is a map $f\colon Q \to Q'$ that preserves $\vee$, $\cdot$ (and $\dagger$). It need not preserve $\top$.

SGF Quantales

An element $f \in Q$ is functional if $f^{\dagger} \cdot f \leq e$ and is a partial unit if both f and $f^{\dagger}$ are functional.

$$\mathcal{I}(Q) = \text{ set of partial units}$$

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An SGF quantale is a unital involutive quantale Q such that

- Q is $\vee$-generated by $\mathcal{I}(Q)$
- $f = ff^\dagger f$ for all $f \in \mathcal{I}(Q)$
- For any $f, g \in \mathcal{I}(Q)$ and $h \in Q_e$ if $f \leq h \cdot \top \vee g$ then $f \leq h \cdot f \vee g$

Frames

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A frame L is a sup-lattice with a meet satisfying

$$x \wedge \left(\bigvee Y\right) = \bigvee_{y \in Y} (x \wedge y)$$

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Definition

Homomorphism of frames = homomorphism of unital quantales between frames

Frames and locales

Definition

$$Frm^{op} = Loc$$

Loc	Frm
X	$\mathcal{O}(X)$
$f\colon X \to Y$	$f^{*}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$

Groupoids

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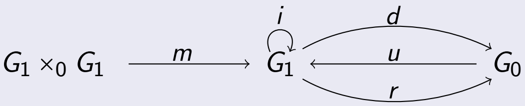
Set groupoids are tuples $G = (G_0, G_1, m, d, r, u, i)$ s.t. G_0 and G_1 are sets, and:



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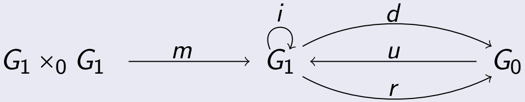
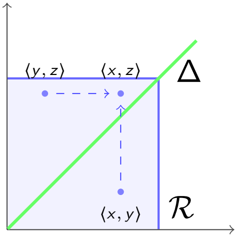


Topological, Localic Groupoids: Groupoids in Top, Loc.

Groupoids and equivalence relations



$$(G_0, G_1, \cdot, d, r, u, ^{-1})$$

$$X = G_0 \qquad \mathcal{R} = G_1$$

$$\langle x, y\rangle \cdot \langle y, z\rangle = \langle x, z\rangle$$

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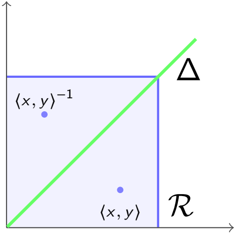

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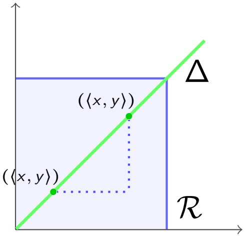

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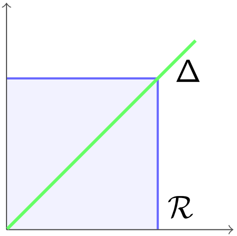

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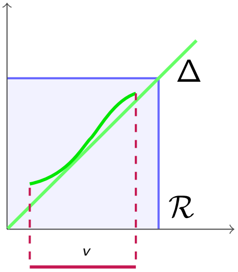

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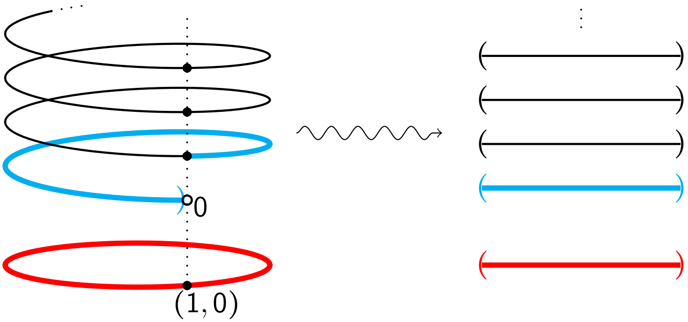
A map $v \colon V \to G_1$ is a local bisection if

- $d \circ v = id_V$
- $r \circ v \colon V \to U$ is a local homeomorphism

Étale vs Non Étale



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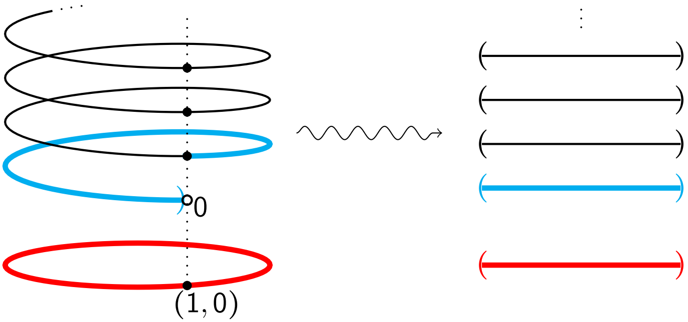

Fact: If G_0 is locally compact then:

- if G is étale, images of local bisections form a basis for the topology of G_1.
- If the topology of G_1 has a basis of images of local bisections, then G is étale.

From groupoids to quantales, first approach

Point set non étale case

G groupoid $\implies \mathcal{P}(G_1)$ can be given the structure of a unital involutive quantale:

$$S \cdot T = \{x \cdot y \mid x \in S, y \in T \text{ and } r(x) = d(y)\}$$

$$S^{\dagger} = \{x^{-1} \mid x \in S\}$$

$$E = \text{ the image of } u\colon G_0 \to G_1$$

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Want: to substitute $\mathcal{P}(G_1)$ with $Sp(G_1) =$ set of sublocales of G_1.

Protin and Resende:

if $\mathcal{G}$ is a localic groupoid such that m^* preserves all meets (iff m^* has left adjoint $m_!$) $\Longrightarrow$ $\mathcal{O}(G_1)$ is quantale, multiplication = composite

$$\mathcal{O}(G_1) \otimes \mathcal{O}(G_1) \xrightarrow{q} \mathcal{O}(G_1 \times_{G_0} G_1) \xrightarrow{m_!} \mathcal{O}(G_1)$$

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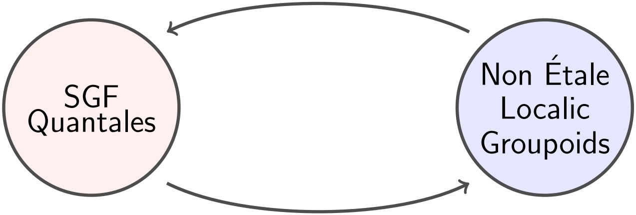
Problem:

m^* need not preserve arbitrary meets.

Localic Non-Étale (Work in progress)



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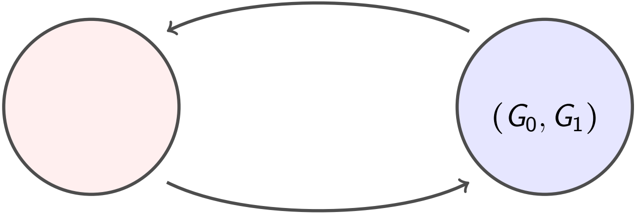

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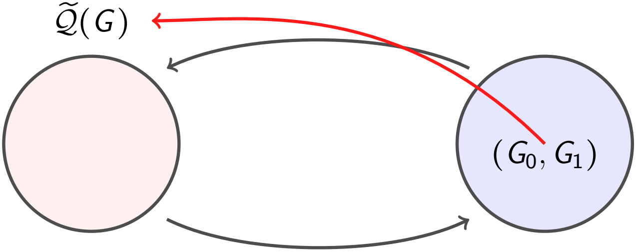

Unital Involutive Quantale

$\tilde{\mathcal{Q}}(G) := \{B \subset \mathcal{O}(G_1) \mid B \text{ is up closed}\}$

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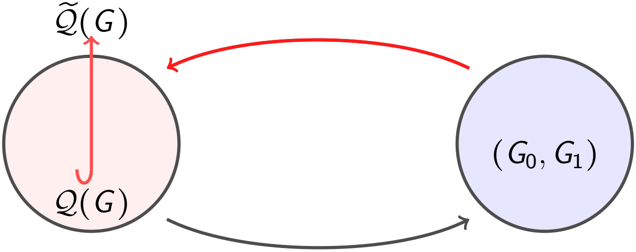

subquantale of $\widetilde{\mathcal{Q}}(G)$
$\bigvee$-generated by locally closed
local bisections

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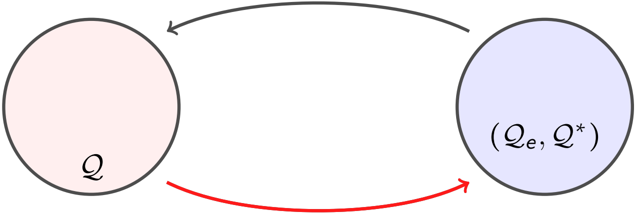

SGF-quantale

$\mathcal{Q}^*$: greatest $\mathcal{Q}' \leq \mathcal{Q}$ s.t.
$\mathcal{Q}_e \leq \mathcal{Q}'$ and
$\mathcal{Q}'$ inverse quantal frame
Conjecture: it exists

From Groupoids to Quantales

2 assumptions:

- $u\colon G_0 \to G_1$ is a closed embedding
- $m\colon G_1 \times_{G_0} G_1 \to G_1$ is a closed map

Construction 1: $\tilde{\mathcal{Q}}(G)$

- Define $\tilde{\mathcal{Q}}(G) = U(G_1) := \{B \subset \mathcal{O}(G_1) \mid B \text{ is upward closed }\}$
- $U(G_1)$ is a complete meet semi lattice, meets = intersections

$$\mathcal{O}(G_1) \hookrightarrow U(G_1)$$
$$a \mapsto (a)\uparrow$$

- $m^*\colon \mathcal{O}(G_1) \to \mathcal{O}(G_1) \otimes_{\mathcal{O}(G_0)} \mathcal{O}(G_1) = \mathcal{O}(G_1 \times_{G_0} G_1)$ can be extended to

$$\bar{m}^*\colon U(G_1) \to U(G_1 \times G_1)$$
$$B \mapsto \bar{m}^*(B) = m^*(B)\uparrow$$

$\tilde{\mathcal{Q}}(G)$ is a unital involutive quantale

$\bar{m}^*$ preserves arbitrary meets $\implies$ has left adjoint $\bar{m}^*_!$

$\exists$ map of sup lattices $U(G_1) \otimes U(G_1) \xrightarrow{q} U(G_1 \times_{G_0} G_1)$

Multiplication:

$$U(G_1) \otimes U(G_1) \xrightarrow{q} U(G_1 \times_{G_0} G_1) \xrightarrow{\bar{m}^*_!} U(G_1)$$

Involution:

$$\dagger \colon \tilde{\mathcal{Q}}(G) \to \tilde{\mathcal{Q}}(G)$$
$$B \mapsto B^\dagger = \{i^*(b) \mid b \in B\}$$

Unit:

$$u(G_0) = a_u \uparrow \text{ for some } a_u \in \mathcal{O}(G_1)$$

Thank you for your attention.

Q a quantale. A nucleus on Q is a closure operator $j\colon Q \to Q$ st

$$j(x)j(y) \leq j(xy) \forall x, y \in Q$$

Open sublocale

L locale, j nucleus on $\mathcal{O}(L)$. The sublocale defined by j is open if j is the nucleus induced by the quotient

$$(-) \wedge a\colon \mathcal{O}(L) \to \downarrow a$$

for some $a \in (\mathcal{O})$

A local bisection $\sigma\colon U \to G_1$ is a section of $d\colon G_1 \to G_0$ over U such that $r \circ \sigma\colon U \to G_0$ is a open embedding, with image an open sublocale $V \subset G_0$.

Open sublocales $U, V \subset G_0$; associate the open sublocale of G_1 defined by

$$G_1(U, V) = d^{-1}(U) \cap r^1(V) \subset G_1.$$

Definition

A local bisection $\sigma \colon U \to G_1$ has domain $U \subset G_0$ and codomain $V \subset G_0$ if $V = r(\sigma(U))$. We say that σ is locally closed in G_1 if $\sigma(U)$ is a closed subspace of $G_1(U, V)$.

Given any locally closed local bisection σ we denote $a_\sigma \in \mathcal{O}(G_1(U,V))$ the uniquely defined element such that the closed subspace $\sigma(U)$ of $G_1(U,V)$ has nucleus image

$$c(a_\sigma) = a_\sigma \uparrow \in U(G_1(U,V))$$

We denote by $Q(U,V) \subset eQ(U,V)$ the join sub suplattice of $\hat{Q}(U,V)$ generated by the upsets of the form $c(a_\sigma) = a_\sigma \uparrow$, associated to locally closed local bisections $\sigma \colon U \to G_1$ with domain U and codomain V .

Fact

From the open embeddings $j\colon G_1(U,V) \to G_1$ one can define the sup-lattice morphisms $j! : \widetilde{Q}(U,V) \to \widetilde{Q}(G)$ for any U, V.

Definition

We define $Q(G)$ as the sub suplattice of $\widetilde{Q}(G)$ that is join-generated by all the images $j!(Q(U,V)) \subset \widetilde{Q}(G)$, for varying $U, V \subset G_0$.