A Hilbert space operator representation of generalized effect algebras of bilinear forms and measures

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Generalized effect algebra

Definition

A partial algebra $(E; \oplus, 0)$ is called a *generalized effect algebra* if $0 \in E$ is a distinguished element and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x, y, z \in E$:

(GEi) $x \oplus y = y \oplus x$, if one side is defined,

(GEii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$, if one side is defined,

(GEiii) $x \oplus 0 = x$,

(GEiv) $x \oplus y = x \oplus z$ implies $y = z$ (cancellation law),

(GEv) $x \oplus y = 0$ implies $x = y = 0$.

In every generalized effect algebra E, a partial binary relation $\leq$ can be defined by

(ED) $x \leq y$ iff there exists an element $z \in E$ such that $x \oplus z$ is defined and $x \oplus z = y$.

A sub-generalized effect algebra

Definition

Let $(E; \oplus, 0)$ be a generalized effect algebra. A subset $Q \subseteq E$ is called a *sub-generalized effect algebra* of E iff

(i) $0 \in Q$,

(ii) if $x, y \in Q$ such that $x \oplus y$ is defined, then $x \oplus y \in Q$,

(iii) if $x, z \in Q$ such that $x \oplus y = z$, then $y \in Q$.

Generalized states

Definition

- A map $\varphi : P_1 \to P_2$ between posets $(P_1; \leq_1), (P_2; \leq_2)$ is *order reflecting* if for any $a, b \in P_1$, $\varphi(a) \leq_2 \varphi(b)$ implies $a \leq_1 b$.
- A set M of isotone maps $\varphi : P_1 \to P_2$ is *order determining* if for any $a, b \in P_1$, $\varphi(a) \leq_2 \varphi(b)$ for all $\varphi \in M$ implies $a \leq_1 b$.
- A *morphism of generalized effect algebras* is a map $\varphi : E_1 \to E_2$ such that $\varphi(0_1) = 0_2$ and whenever $a \oplus_1 b$ is defined, then $\varphi(a \oplus_1 b) = \varphi(a) \oplus_2 \varphi(b)$, for any $a, b \in E_1$.
- A *generalized state* is a morphism $\varphi : E \to \mathbb{R}^+$, $\mathbb{R}^+$ equipped with the usual sum.

A generalized effect algebra of operators on $\mathcal{H}$

$Lin(\mathcal{H})$ - linear operators on a sub-space of $\mathcal{H}$.

Theorem (Riečanová Z., Zajac M., Pulmannová S., 2011)

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let $\mathcal{V}(\mathcal{H}) \subseteq Lin(\mathcal{H})$ be the set of linear operators such that

$$\mathcal{V}(\mathcal{H}) := \{A \mid A \geq \mathbf{0}, \overline{D(A)} = \mathcal{H} \text{ and if } A \text{ is bounded, then } D(A) = \mathcal{H}\}$$

Let $\oplus$ be a partial operation on $\mathcal{V}(\mathcal{H})$ defined by

- *for $A, B \in \mathcal{V}(\mathcal{H})$, $A \oplus B$ is defined if and only if A or B is bounded or $D(A) = D(B)$ and then $A \oplus B = A + B$.*

Then $(\mathcal{V}(\mathcal{H}); \oplus, \mathbf{0})$ is a generalized effect algebra.

Bilinear forms on $\mathcal{H}$

Let $D(t) \subseteq \mathcal{H}$ be a linear subspace of $\mathcal{H}$,

- a *bilinear form* is a map $t : D(t) \times D(t) \to \mathbb{C}$ such as t is additive in both arguments and $(\alpha x, \beta y) = \alpha\overline{\beta}(x, y)$ for all $\alpha, \beta \in \mathbb{C}$, $x, y \in D(t)$, where $\overline{\beta}$ is the complex conjugation of β.
- t is *symmetric* if $t(x, y) = \overline{t(y, x)}$, *positive* if $t(x, x) \geq 0$ for all $x \in D(t)$, *bounded* if there exists $c \in \mathbb{R}$, such that $t(x, x) \leq c$, for all $x \in D(t), \|x\| = 1$.
- Given a positive bilinear form t, we can equip its domain $D(t)$ with an inner product $(x, y)_t := t(x, y) + (x, y)$. Whenever $D(t)$ with $(x, y)_t$ is a Hilbert space, we call t *closed.*
- A bilinear form t is *closable* if it has some closed extension.

Bounded bilinear forms

Proposition

There is a one-to-one correspondence between bounded linear operators and bounded bilinear forms on $\mathcal{H}$ given by $t(x, y) = (Ax, y)$ for some $A \in \mathcal{B}(\mathcal{H})$ and all $x, y \in \mathcal{H}$.

On the set of all bilinear forms we can define a *usual sum* $t + s$ for each t, s on $D(t + s) := D(t) \cap D(s)$ by $(t + s)(x, y) := t(x, y) + s(x, y)$ for all $x, y \in D(t) \cap D(s)$, and the multiplication by a scalar $\alpha \in \mathbb{C}$ by $(\alpha t)(x, y) := \alpha t(x, y)$ for $x, y \in D(\alpha t) := D(t)$.

A general case of bilinear forms

Theorem (Simon B., 1978)

Let t be a densely defined positive symmetric bilinear form on a Hilbert space $\mathcal{H}$. Then there exist two positive symmetric bilinear forms t_r and t_s such that $D(t) = D(t_r) = D(t_s)$ such that

$$t = t_r + t_s, \tag{1}$$

where t_r is the largest closable bilinear form less than t in the ordering $\preceq$.

Proposition

For any positive operator $A : D(A) \to \mathcal{H}$, the induced bilinear form $t(x, y) = (Ax, y)$ on $D(t) = D(A)$ is closeable (that is $t_s = 0$).

A generalized effect algebra of bilinear forms

$\mathcal{PBF}$:= positive bilinear forms

Theorem (Dvurečenskij A., J.J., 2013)

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let us define the set of bilinear forms

$$\mathcal{V}_f(\mathcal{H}) = \{t \mid t \in \mathcal{PBF}, \overline{D(t)} = \mathcal{H} \text{ and if } t \text{ is bounded, then } D(t) = \mathcal{H}\}.$$

Let us define a partial operation $\oplus$ on $\mathcal{V}_f(\mathcal{H})$ by

- *for $t, s \in \mathcal{V}_f(\mathcal{H})$, $t \oplus s$ is defined if and only if t or s is bounded or $D(t) = D(s)$ and then $t \oplus s = t + s$.*

Then $(\mathcal{V}_f(\mathcal{H}); \oplus, o)$ is a generalized effect algebra.

A sub-GEA with fixed domain

Theorem (Dvurečenskij A., J.J., 2013)

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $D \subseteq \mathcal{H}$ its dense linear subspace. Let us define the set $\mathcal{V}_{fD}(\mathcal{H}) \subseteq \mathcal{V}_f(\mathcal{H})$ by

$$\mathcal{V}_{fD}(\mathcal{H}) = \{t \in \mathcal{V}_f(\mathcal{H}) \mid t \text{ is bounded, or } D(t) = D\}.$$

Then $\mathcal{V}_{fD}(\mathcal{H})$ is a sub-generalized effect algebra of $(\mathcal{V}_f(\mathcal{H}); \oplus, o)$,. Moreover, operation $\oplus_{|\mathcal{V}_{fD}(\mathcal{H})}$, is total on $\mathcal{V}_{fD}(\mathcal{H})$ and $(\mathcal{V}_{fD}(\mathcal{H}); \oplus_{|\mathcal{V}_{fD}(\mathcal{H})}, o)$, is monotone Dedekind upwards and downwards σ-complete.

Po-groups

Definition

An abelian group $(G;+,0)$ is called *po-group* with partial order $\leq$ if for all $x,y,z \in G$, $x \leq y$ implies $x+z \leq y+z$ (we write $(G;+,\leq,0)$). Then $\mathrm{Pos}(G) := \{x \in G \mid 0 \leq x\}$.

Definition

Let $(G_1;+_1,\leq_1,0_1)$ and $(G_2;+_2,\leq_2,0_2)$ be a po-groups. A *homomorphism of po-groups* $f : G_1 \to G_2$ is an order preserving homomorphism of groups. We call f an *order embedding* if $f(x) \leq_2 f(y)$ implies $x \leq_1 y$. An $\mathbb{R}$-*map* $f : G_1 \to G_2$ is a homomorphism of po-group such that $G_2 = \mathbb{R}$ (with the usual sum and order on reals).

Po-groups of linear operators

Proposition (Chajda I., Paseka J., Lei Q., 2013)

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $D \subseteq \mathcal{H}, \overline{D} = \mathcal{H}$. Let

$$Lin_D(\mathcal{H}) = \{A : D \to \mathcal{H} \mid A \text{ is a linear operator defined on } D\}$$

Then $(Lin_D(\mathcal{H}); +, \leq, \mathbf{0})$ is a partially ordered abelian group where, $+$ is the usual sum of operators defined on D and $\leq$ is defined for all $A, B \in Lin_D(\mathcal{H})$ by $A \leq B$ iff $B - A$ is positive.
Moreover, let

$$Symm_D(\mathcal{H}) = \{A \in Lin_D(\mathcal{H}) \mid A \text{ is a symmetric linear operator}\}.$$

Then $(Symm_D(\mathcal{H}); +, \leq, \mathbf{0})$ is a partially ordered subgroup of the partially ordered group $(Lin_D(\mathcal{H}); +, \leq, \mathbf{0})$.

Po-groups of bilinear forms

Theorem

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $D \subseteq \mathcal{H}, \overline{D} = \mathcal{H}$ its dense sub-space. Then the set

$$\mathcal{F}_D(\mathcal{H}) = \{t : D(t) \times D(t) \to \mathbb{C} \mid t \textit{ is a bilinear form}, D(t) = D\}$$

with operation $+$ being the usual pointwise sum of mappings, and a partial order $\leq$ defined by $r \leq s$ iff $s - r$ is a positive bilinear form, forms a partially ordered abelian group $(\mathcal{F}_D(\mathcal{H}); +, \leq, o)$.
Moreover, let

$$\mathcal{S}_D(\mathcal{H}) = \{t \mid t \in \mathcal{F}_D(\mathcal{H}), t \textit{ is a symmetric bilinear form}\}.$$

Then $(\mathcal{S}_D(\mathcal{H}); +, \leq, o)$ is a partially ordered abelian subgroup of $\mathcal{F}_D(\mathcal{H})$.

Positive cones in po-groups are GEA's

Remark

In every partially ordered abelian group $(G; +, \leq, 0)$, *a set of all positive elements* $Pos(G)$ *with restricted operation* $+_{|Pos(G)}$ *forms a generalized effect algebra* $(Pos(G); +_{|Pos(G)}, 0)$. *The restriction* $f_{|Pos(G)}$ *of any* $\mathbb{R}$*-map* f *is a generalized state on a generalized effect algebra* $Pos(G)$.

Remark

A positive elements $Pos(\mathcal{S}_D(\mathcal{H}))$ *of a po-group* $\mathcal{S}_D(\mathcal{H})$ *forms an isomorphic generalized effect algebra to* $(\mathcal{V}_{fD}(\mathcal{H}); \oplus_{|\mathcal{V}_{fD}(\mathcal{H})}, o)$.

Representation theorem for EAs

Let $(E; \oplus, 0, 1)$ be an effect algebra with an order determining set $\mathcal{M}$ of states. Let

$$l_2(\mathcal{M}) = \{(x_\omega)_{\omega \in \mathcal{M}} \mid x_\omega \in \mathbb{C}, \sum_{\omega \in \mathcal{M}} |x_\omega|^2 < \infty\}$$

be a Hilbert space with a usual inner product

$$\langle (x_\omega)_{\omega \in \mathcal{M}}, (y_\omega)_{\omega \in \mathcal{M}} \rangle = \sum_{\omega \in \mathcal{M}} \overline{x}_\omega \cdot y_\omega$$

We have $\mathcal{E}(l_2(\mathcal{M})) = [\mathbf{0}, I] \subseteq \mathcal{B}^+(l_2(\mathcal{M}))$.

Theorem (Riečanová Z., Zajac M.)

An effect algebra $(E; \oplus, 0, 1)$ has a Hilbert space effect-representation φ iff there exists an order determining set $\mathcal{M}$ of states on E.

Namely, such $\varphi : E \to \mathcal{E}(l_2(\mathcal{M}))$ that for every $a \in E$ the image $\varphi(a)$ is the operator $A \in \mathcal{E}(l_2(\mathcal{M}))$ defined by $A\mathbf{x} = (\omega(a)x_\omega)_{\omega \in \mathcal{M}}$, for all $\mathbf{x} \in l_2(\mathcal{M})$.

Representation theorem for po-groups

Theorem (Chajda I., Paseka J., Lei Q., 2013)

For every partially ordered abelian group G, the following conditions are equivalent.

- *There exists an order determining set M of $\mathbb{R}$-maps on G,*
- *there exists an order embedding of G to the symmetric operators on a dense subspace of a Hilbert space,*
- *there exists a set T and an order embedding of G to $\mathbb{R}^T$.*

An embedding of a po-group of bilinear forms

Lemma

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $D \subseteq \mathcal{H}, \overline{D} = \mathcal{H}$ its dense sub-space. Then, for every $x \in D$, a map $\omega_x : \mathcal{S}_D(\mathcal{H}) \to \mathbb{R}$ given by $\omega_x(t) := t(x,x)$ is an $\mathbb{R}$-map and the set $M = \{\omega_x \mid x \in D, ||x|| = 1\}$ is an order determining set of $\mathbb{R}$-maps.

Theorem

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $D \subseteq \mathcal{H}, \overline{D} = \mathcal{H}$ its dense sub-space. Then there exists an order embedding from the partially ordered abelian group $(\mathcal{S}_D(\mathcal{H}); +, \leq, o)$ of symmetric bilinear forms on D into the partially ordered abelian group of symmetric linear operators $(Symm(M); +, \leq, \mathbf{0})$ on the dense subspace of the Hilbert space $l_2(M)$.

Corollary for GEAs

Corollary

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $D \subseteq \mathcal{H}, \overline{D} = \mathcal{H}$ its dense sub-space. Then there exists an order determining set M of generalized states on $Pos(\mathcal{F}_D(\mathcal{H}))$ and an order reflecting morphism from the generalized effect algebra $(Pos(\mathcal{F}_D(\mathcal{H})); +_{|Pos(\mathcal{F}_D(\mathcal{H}))}, o)$ of positive bilinear forms on D into the generalized effect algebra of positive linear operators $(Pos(Symm(M)); +_{|Pos(Symm(M))}, \mathbf{0})$ on the dense subspace of the Hilbert space $l_2(M)$.

Measures on $L(\mathcal{H})$

Definition

A mapping $m : L(\mathcal{H}) \to [0, \infty]$ is said to be a *finitely additive measure* if (i) $m(\mathrm{sp}(\mathbf{0})) = 0$, and (ii) $m(M \vee N) = m(M) + m(N)$ whenever $M \perp N$, $M, N \in L(\mathcal{H})$.

Definition

A finitely additive measure m on $L(\mathcal{H})$ is said to be (i) *regular* if

$$m(M) = \sup\{m(P) \mid P \subseteq M, P \in L(\mathcal{H}), \dim P < \infty\}, \quad M \in L(\mathcal{H}),$$

(ii) $\mathcal{P}(\mathcal{H})_1$-*bounded* if $\sup\{m(\mathrm{sp}(x)) \mid x \in D(m)\} < \infty$, where

$$D(m) := \{x \in \mathcal{H} \mid m(\mathrm{sp}(x)) < \infty\} \cup \{\mathbf{0}\},$$

(iii) satisfies *L-S density property* if $\overline{D(m)} = \mathcal{H}$ and there is a two-dimensional subspace Q of $\mathcal{H}$ such that $m(Q) < \infty$.

GEA of measures

Theorem

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let $\mathrm{Reg}_f(\mathcal{H})$ be the set of regular finitely additive measures m on $L(\mathcal{H})$ with the L-S density property such that if m is $\mathcal{P}_1(\mathcal{H})$-bounded, then $D(m) = \mathcal{H}$. Let us define a partial operation $\oplus$ on $\mathrm{Reg}_f(\mathcal{H})$: For $m_1, m_2 \in \mathrm{Reg}_f(\mathcal{H})$, $m_1 \oplus m_2$ is defined if and only if m_1 or m_2 is $\mathcal{P}_1(\mathcal{H})$-bounded or $D(m_1) = D(m_2)$ and then $m_1 \oplus m_2 := m_1 + m_2$. Then $(\mathrm{Reg}_f(\mathcal{H}); \oplus, o)$ is a generalized effect algebra.

An extension of Gleason's Theorem

Theorem (Dvurečenskij A.)

Let $\mathcal{H}$ be a complex Hilbert space.
(1) *Let t be a positive bilinear form such that $D(t)$ is dense in H. Then the mapping $m_t : L(\mathcal{H}) \to [0, \infty]$ given by*

$$m_t(M) = \begin{cases} \mathrm{tr}(t \circ P_M) & \text{if } t \circ P_M \in \mathrm{Tr}(\mathcal{H}), \\ \infty & \text{otherwise}, \end{cases} \tag{6.2}$$

is a regular finitely additive measure with the L-S density property.
(2) *Let m be a regular finitely additive measure with the L-S density property on $L(\mathcal{H})$, $\dim \mathcal{H} \neq 2$. Then there exists a unique bilinear form t with domain $D(t) = D(m)$ such that (6.2) holds.*

Representation of GEA's of measures by linear operators

Theorem

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and let $\overline{D} \in \mathcal{H}$ be a linear subspace of $\mathcal{H}$. Let $\mathrm{Reg}_{fD}(\mathcal{H})$ be the set of regular finitely additive measures m on $\mathcal{L}(\mathcal{H})$ with the L-S density property such that if m is $\mathcal{P}_1(\mathcal{H})$-bounded, then $D(m) = \mathcal{H}$, in other case $D(m) = D$. Then $\mathrm{Reg}_{fD}(\mathcal{H})$ forms a sub-generalized effect algebra of $(\mathrm{Reg}_f(\mathcal{H}); \oplus, o)$. Moreover, there exists an order determining set M of generalized states on $\mathrm{Reg}_{fD}(\mathcal{H})$ and an embedding from the generalized effect algebra $(\mathrm{Reg}_{fD}(\mathcal{H}); +_{\mathrm{Reg}_{fD}(\mathcal{H})}, o)$ into the generalized effect algebra of positive linear operators $(Pos(Symm(M)); +_{|Pos(Symm(M))}, \mathbf{0})$ on the dense subspace of the Hilbert space $l_2(M)$.

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Thank you for your attention!