On $\mathbf{\Sigma}^0_2(\kappa)$ Relations and Elementary Embeddability at Uncountable Cardinals

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Model theoretic spectrum functions

Let T be a first order theory.

Classification problem: find a structure theorem for the models of T or show that such a theorem can not exist.

Spectrum functions

For each cardinal κ, let $I(T, \kappa)$= the number of κ-sized models of T up to isomorphism.

For countable theories: Shelah; Hart, Hrushovski and Laskowski.

Variants of spectrum functions

- Replace the role of isomorphisms by that of other natural notions: embeddings, elementary embeddings, maps that preserve formulas in some infinitary logics ...
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  "How many" pairwise non H-elementarily embeddable models with domain κ does T have?

  Similar questions for H-embeddability, H-isomorphism,...
- Consider classes of models not defined by a first order theory.

Generalized Baire spaces

The domain of the κ-Baire space is ${}^{\kappa}\kappa$. Its topology is given by the basic open sets

$$N_p = \{f \in {}^{\kappa}\kappa : p \subseteq f\},$$

where $p : \alpha \longrightarrow \kappa$ for some $\alpha < \kappa$.

Generalized descriptive set theory: studies the κ-Borel structure of ${}^{\kappa}\kappa$.

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- κ-Borel sets: close the family of open subsets under intersections and unions of size $\leq \kappa$ and complementation;
- $\mathbf{\Sigma}^0_2(\kappa)$-sets: unions of at most κ many closed sets;
- $\mathbf{\Pi}^0_2(\kappa)$-sets: intersections of at most κ many open sets;
- κ-analytic sets: continuous images of κ-Borel sets.

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Often very different from the countable case; many properties depend on which additional set theoretical hypotheses are assumed.
One motivation: topological study of uncountable structures, their classification.

A dichotomy for $\mathbf{\Sigma}^0_2(\kappa)$ binary relations

Theorem (Sz., Väänänen)

Assume the hypotheses $\mathrm{I}^-(\kappa)$ and $\Diamond_\kappa$. Suppose R is a $\mathbf{\Sigma}^0_2(\kappa)$ binary relation on a κ-analytic subset X of ${}^\kappa\kappa$.

Either all independent sets are of size $\leq \kappa$, or there is a κ-perfect independent set,

i.e., there is a continuous embedding

$$f : {}^\kappa 2 \to X \text{ such that } x \neq y \text{ implies } (f(x), f(y)) \notin R.$$

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- The theorem does not follow from ZFC alone (since other set theoretical hypotheses consistent with ZFC imply it can not hold).
- $\mathrm{I}^-(\kappa)$ is a modification of the hypothesis $\mathrm{I}(\kappa)$ found in literature and states:

  > *there exists a* κ^+*-complete normal ideal* $\mathcal{I}$ *on* κ^+ *and a dense subset* $K \subseteq \mathcal{I}^+$ *such that every descending sequence of elements of* K *of length* $< \kappa$ *has a lower bound in* K.

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- $\diamondsuit_\kappa$ is a combinatorial hypothesis well-known in set theory:

  There exist sets $S_\alpha \subseteq \alpha$ $(\alpha < \kappa)$ *such that for all* $S \subseteq \kappa$ *the set* $\{\alpha < \kappa : S \cap \alpha = S_\alpha\}$ *is a stationary subset of* κ*.*

- The consistency of "$\mathrm{I}^-(\kappa)$ and $\diamondsuit_\kappa$" follows from a well-known large cardinal axiom (the existence of a measurable cardinal $\lambda > \kappa$).
- $\diamondsuit_\kappa$ does not have to be assumed when κ is inaccessible.

Cylindric algebras

A cylindric algebra of dimension κ is an algebraic structure

$$\mathcal{A} = \langle A, \wedge, -, c_i, d_{ij} \rangle_{i,j<\kappa}$$

such that

- $\langle A, \wedge, - \rangle$ is a Boolean algebra;
- c_i is a unary operation; corresponds to $\varphi \mapsto \exists v_i \varphi$;
- d_{ij} is a constant; corresponds to the formula $v_i = v_j$;
- certain axioms are satisfied.

Cylindric algebras and models

L a first order language with κ variables, T an L-theory.

Let $\mathcal{L}(T)$ be the Lindenbaum-Tarski algebra of T;
elements: equivalence classes of L-formulas of

$$\varphi(\bar{v}) \equiv_T \psi(\bar{v}) \iff T \models \forall \bar{v}\big[\varphi(\bar{v}) \leftrightarrow \psi(\bar{v})\big]$$

operations: Boolean operations and

$$c_i(\varphi/{\equiv_T}) = (\exists v_i \varphi)/{\equiv_T}, \qquad d_{ij} = (v_i = v_j)/{\equiv_T} \ .$$

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There is a one-one correspondence between

- models of T;
- homomorphisms from $\mathcal{L}(T)$ onto certain cylindric *set* algebras.

Representation theory

Proposition (Sági, Sz.)

There is a one-one correspondence between

- *the models of* T *with domain* κ *and*
- *certain ultrafilters of (the Boolean reduct of)* $\mathcal{L}(T)$*, i.e., a certain set* $\mathcal{H}(T)$ *of Boolean homomorphisms from* $\mathcal{L}(T)$ *to* $\underline{2}$*.*

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$\mathbf{\Sigma}^1_1(L)$ consists of formulas of the form $\exists\overline{R}\psi(\overline{R})$ where $\overline{R}$ is a set of "new" relation symbols.

In the case that $T \subseteq \mathbf{\Sigma}^1_1(L)$, the set of models of T with domain κ will correspond to a κ-analytic subset of the κ-Baire space.

Characterization of elementary embeddings

Let $\rho \in {}^{\kappa}\kappa$. Suppose the models $\mathcal{M}_0$ and $\mathcal{M}_1$ of T with domain κ correspond to the homomorphisms $u_0, u_1 \in \mathcal{H}(T)$.

Proposition (Sági, Sz)

- *(If ρ is injective, then)* $\rho : \mathcal{M}_0 \preccurlyeq \mathcal{M}_1$ *iff* $u_0 = u_1 \circ s_\rho$.
- *If ρ is bijective, then* $\rho : \mathcal{M}_0 \cong \mathcal{M}_1$ *iff* $u_0 = u_1 \circ s_\rho$.

s_ρ is the *substitution operation* correlated to ρ;

$$s_\rho\big(\varphi(v_1, \ldots, v_n)/{\equiv_T}\ \big) = \varphi(v_{\rho(1)}, \ldots, v_{\rho(n)})/{\equiv_T}$$

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Corollary

For $H \subseteq {}^{\kappa}\kappa$, the set $\{\langle \rho, \mathcal{M}_0, \mathcal{M}_1 \rangle \mid H \ni \rho : \mathcal{M}_0 \preccurlyeq \mathcal{M}_1\}$ corresponds to a closed subset of $H \times \mathcal{H}(T) \times \mathcal{H}(T)$.

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A set F of $L_{\kappa^+\kappa}$-formulas is a fragment if $|F| = \kappa$ and

- F contains all the atomic formulas,
- F is closed under negation, taking subformulas, and substitution of variables.

Examples: the infinitary logics $L_{\kappa\omega}$, $L_{\kappa\kappa}$, the set of quantifier-free formulas ($\rightsquigarrow$ embeddability), the set L^n of formulas with at most n variables,...

By considering the algebras $\mathcal{F}(T) = \langle F/{\equiv_T}, \wedge, \neg, s_\tau, d_{ij}\rangle_{\tau\in{}^\kappa\kappa, i,j\in\kappa}$ and similarly to previous arguments, we obtain:

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By considering the algebras $\mathcal{F}(T) = \langle F/{\equiv_T}, \wedge, \neg, s_\tau, d_{ij}\rangle_{\tau\in{}^\kappa\kappa, i,j\in\kappa}$ and similarly to previous arguments, we obtain:

- if $T \subseteq F$, then its set of models with domain κ corresponds to a κ-Borel subset $\mathcal{H}(T)$ of the ${}^\kappa\kappa$;
- when $\psi \in \boldsymbol{\Sigma}^1_1(L_{\kappa^+\kappa})$ is arbitrary, $\mathcal{H}(\psi)$ is a κ-analytic subset of the κ-Baire space.
- $\rho : \mathcal{M}_0 \longrightarrow \mathcal{M}_1$ preserves the formulas in F iff $u_0 = u_1 \circ s_\rho$.

(F, H)-elementary embeddability

Let $\mathcal{A}, \mathcal{B}$ be models with domain κ, let $H \subseteq {}^{\kappa}\kappa$ and let F be a fragment.

Definition

$\mathcal{A}$ is (F, H)-*elementarily embeddable* into $\mathcal{B}$ if there exists $f \in H$ that preserves the formulas in F,
(i.e, for all $\varphi \in F$ and $\bar{a} \in {}^{<\kappa}\kappa$ we have $\mathcal{A} \models \varphi[\bar{a}]$ iff $\mathcal{B} \models \varphi[f(\bar{a})]$.)

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Special case: H-embeddability; H-elementary embeddability; ...
H-isomorphism, when H is a subgroup of $Sym(\kappa)$.

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(F, H)-elementary embeddability is a projection of this relation. Therefore it is a $\mathbf{\Sigma}^0_2(\kappa)$ binary relation on $\mathcal{H}(T)$, when H is a K_κ subset of ${}^\kappa\kappa$.

Definition

A subset H of ${}^\kappa\kappa$ is

- κ-compact, if any open cover of H has a subcover of size $< \kappa$;
- K_κ, if it is the union of at most κ many κ-compact subsets.

A dichotomy for κ-sized models

Corollary (Sz., Väänänen)

Assume $\mathrm{I}^-(\kappa)$ *and either* $\diamondsuit_\kappa$ *or that* κ *is inaccessible. Let* H *be a* K_κ *subset of* κ*-Baire space,* F *a fragment of* $L_{\kappa^+\kappa}$ *and* ψ *a sentence of* $L_{\kappa^+\kappa}$*.*

If ψ *has* κ^+ *many pairwise non* (F,H)*-elementarily embeddable models with domain* κ*, then there is a* κ*-perfect set of such models.*

Thank you for your attention!