A topological duality for posets

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TACL 2015

Ischia, June 26, 2015.

Introduction

In 2014 M. A. Moshier and P. Jipsen published:

• *Topological Duality and Lattice Expansions Part I: A Topological Construction of Canonical Extensions*, Algebra Universalis 71 (2014), and its sequel

• *Topological Duality and Lattice Expansions Part II: Lattice Expansions with Quasioperators*, Algebra Universalis 71 (2014).

In the first paper, a topological duality for the category of bounded lattices (with morphisms the lattice homomorphisms) is introduced.

The duality for objects is used to provide a topological proof of the existence of the canonical extension of a lattice, a notion introduced by M. Gehrke and J. Harding in *Bounded lattice expansions*, Journal of Algebra 238 (2001).

To obtain the duality, M. A. Moshier and P. Jipsen develop first a topological duality for the category of meet-semilattices with top element (with meet-and-top-preserving maps as the morphisms).

The objects of the dual category are the HSM-spaces (Hofmann-Mislove-Stralka spaces): sober spaces where the set of compact open filters (w.r.t. the specialization order) is a basis and is closed under finite intersections.

The morphisms are the continuous maps with the property that the inverse image of a compact open filter is a compact open filter.

Using inspiration on this duality we managed to provide:

- a topological duality for a category with objects the posets.
- use it to obtain a topological proof of the existence of the canonical extension of a poset, as defined by M. Gehrke, M. Dunn and A. Palmigiano in *Canonical extensions and relational completeness of some substructural logics*, The Journal of Symbolic Logic 70 (2005).

Outline of the talk

- The duality for the objects.
- The duality for the morphisms.
- The duality for posets related to the HSM-spaces.
- A topological proof of the existence of the canonical extension of a poset.

Towards the dual of a poset

Let $P = \langle P, \leq \rangle$ be a poset.

A set $F \subseteq P$ is a filter of P if it is a nonempty down-directed up-set.

$\mathsf{F}(P)$ will denote the set of filters of P and also the poset $\langle \mathsf{F}(P), \subseteq \rangle$.

We define the topological space

$$X_P := \langle \mathsf{F}(P), \tau_P \rangle$$

obtained by taking the Scott topology of the poset $\mathsf{F}(P)$.
(This topology consists of the up-sets of this poset which are inaccessible w.r.t. joins of up-directed subsets of $\mathsf{F}(P)$).

- The spezialization order of X_P is the inclusion relation on $\mathsf{F}(P)$.

- The space X_P has as a basis the family $\{\varphi_P(a) : a \in P\}$ of

$$\varphi_P(a) := \{F \in \mathsf{F}(P) : a \in F\} = \{F \in \mathsf{F}(P) : \uparrow a \subseteq F\}.$$

This holds because:

- If $\mathcal{D}$ is an up-directed family of filters, then its join exists and is $\bigcup \mathcal{D}$.
- If U is a Scott-open set, and $F \in U$, then $\{\uparrow a : a \in F\}$ is up-directed and $F = \bigcup\{\uparrow a : a \in F\}$; therefore there exists $a \in F$ such that $\uparrow a \in U$, which implies $F \in \varphi_P(a) \subseteq U$.

- The space X_P is T_0.

Given a T_0-space X, a filter of X is a filter of the poset $\langle X, \sqsubseteq \rangle$, where $\sqsubseteq$ is the specialization order.

An open filter of X is a filter of X which is open.

A compact open filter of X is an open filter of X which is a compact set.

$\mathrm{KOF}(X)$ denotes the family of all compact open filters of X.

Fact: $$\mathrm{KOF}(X) = \{\uparrow x : x \in \mathrm{Fin}(X)\},$$

where $\mathrm{Fin}(X)$ is the set of finite points of X. A point $x \in X$ is finite if $\uparrow x$ is open.

Some more facts on X_P:

- $\varphi_P(a)$ is a compact open filter of X_P.
- The compact open filters of X_P are exactly the sets $\varphi_P(a)$.

Thus,
$$\mathrm{KOF}(X_P) = \{\varphi_P(a) : a \in P\}.$$

- φ_P establishes an isomorphism between P and $\mathrm{KOF}(X_P)$.

Pst-spaces

Definition

A topological space $X = \langle X, \tau \rangle$ is a Pst-space if

1. it is a sober space,
2. the set $\mathrm{KOF}(X)$ is a basis for τ.

Theorem

Let $X = \langle X, \tau \rangle$ be a topological space. Then, X is a Pst-space if and only if

(*i*) *X is a Scott space;*

(*ii*) $\mathrm{KOF}(X)$ *is a base for τ;*

(*iii*) *every up-directed subset of X (w.r.t. $\sqsubseteq$) has a join.*

Thus applying the theorem we have that:

Theorem

For every poset P, the space X_P is a Pst-space.

The dual poset of a Pst-space

Let X be a Pst-space.

The poset $P_X = \langle \mathrm{KOF}(X), \subseteq \rangle$ will be the dual poset of X.

Theorem

If X is a Pst-space, then X is homeomorphic to X_{P_X}. The homomorphism is established by the map $\theta_X : X \to X_{P_X}$ defined by

$$\theta_X(x) := \{U \in \mathrm{KOF}(X) : x \in U\}.$$

Duals of morphisms

Let P, Q be posets. Let $j : P \to Q$ be an order-preserving map that in addition satisfies that the inverse image of a filter of Q is a filter of P.

The map $\Gamma(j) : X_Q \to X_P$ defined by

$$\Gamma(j)(F) := j^{-1}[F]$$

for every $F \in \mathsf{F}(Q)$

- is continuous and
- $\Gamma(j)^{-1}[U] \in \mathsf{KOF}(X_Q)$, for every $U \in \mathsf{KOF}(X_P)$.

Definition

Let X and Y Pst-spaces. A map $f : X \to Y$ is *F-continuous* if it is continuous and for every $U \in \mathsf{KOF}(Y)$, $f^{-1}[U] \in \mathsf{KOF}(X)$.

If $f : X \to Y$ is F-continuous, then the map $\Delta(f) : P_Y \to P_X$ defined by

$$\Delta(f)(U) = f^{-1}[U]$$

is order-preserving and the inverse image of a filter of P_X is a filter of P_Y.

The categories

We consider the following categories.

- The category of posets with morphisms the order-preserving maps between them with the property that the inverse image of a filter is a filter.
- The category of Pst-spaces with morphisms the F-continuous maps.

These two categories are dually equivalent by the functors that arise from the previous definitions.

The natural transformations

They are the families

- $\{\varphi_P : P \text{ is a poset}\}$,
- $\{\theta_X : X \text{ is a Pst-space}\}$

because the diagrams

$$\begin{tikzcd} P \arrow[r, "j"] \arrow[d, "\varphi_P"'] & Q \arrow[d, "\varphi_Q"] \\ \mathrm{KOF}(X_P) \arrow[r, "\Delta(\Gamma(j))"'] & \mathrm{KOF}(X_Q) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\theta_X"'] & Y \arrow[d, "\theta_Y"] \\ X_{P_X} \arrow[r, "\Gamma(\Delta(f))"'] & X_{P_Y} \end{tikzcd}$$

commute.

Pst-spaces and HSM-spaces

Let P be a poset which is a meet-semilattice. Then X_P satisfies the additional condition that if $U, V \in \mathsf{KOF}(X_P)$, then $U \cap V \in \mathsf{KOF}(X_P)$.

The meet-preserving maps between meet-semilattices are exactly the order preserving maps such that the inverse image of a filter is a filter.

If P is a meet-semilattice with top element, then $\mathsf{KOF}(X_P)$ is closed under intersections of arbitrary finite subsets (in particular $X_P \in \mathsf{KOF}(X_P)$). Thus X_P is an HSM-space. Conversely if X is an HSM-space, then $\mathsf{KOF}(X)$ is a meet-semilattice with top element.

HSM-spaces are spectral spaces.

Pst-spaces need not be spectral, even if we restrict to the compact ones.

The canonical extension of a poset

Let $P = \langle P, \leq \rangle$ be a poset. A completion of P is a pair (e, Q) where Q is a poset that is a complete lattice and e is an order-embedding from P to Q.

Let (e, Q) be a completion of P. An element $a \in Q$ is closed if there exists a filter F of P such that $a = \bigwedge e[F]$. It is open if there exists and ideal I of P (an up-directed down-set) such that $a = \bigvee [I]$.

A completion (e, Q) of P is a canonical extension if

1. the set of open elements of Q is meet-dense in Q and the set of closed elements is join-dense in Q.
2. for every filter F of P and every ideal I of P,

$$\text{if } \bigwedge e[F] \leq \bigvee e[I], \text{ then } F \cap I \neq \emptyset.$$

Theorem

Every poset has a (unique) canonical extension.

The topological duality for posets provides a proof of the existence of the canonical extension of a poset.

Let X be a Pst-space. Let $\mathrm{OF}(X)$ denote the family of all open filters of X and let $\mathrm{Fsat}(X)$ be the closure system on X generated by $\mathrm{OF}(X)$. The elements of $\mathrm{Fsat}(X)$ are called F-saturated sets.

We have the complete lattice $\langle \mathrm{Fsat}(X), \bigcap, \bigvee \rangle$,
where $\bigvee \mathcal{A} = \mathrm{fsat}(\bigcup \mathcal{A})$ for each $\mathcal{A} \subseteq \mathrm{Fsat}(X)$, taking fsat(.) to be the closure operator associated with $\mathrm{Fsat}(X)$, i.e. for every $A \subseteq X$,

$$\mathrm{fsat}(A) = \bigcap \{F \in \mathrm{OF}(X) : A \subseteq F\}.$$

It holds that $\mathrm{KOF}(X) \subseteq \mathrm{OF}(X) \subseteq \mathrm{Fsat}(X)$.
So, the lattice $\mathrm{Fsat}(X)$ is a completion of the poset $P_X = \langle \mathrm{KOF}(X), \subseteq \rangle$.

Theorem

Let P be a poset. The complete lattice $\mathrm{Fsat}(X_P)$ *is the canonical extension of P, with embedding φ_P.*

How do the open and closed elements of $\mathrm{Fsat}(X_P)$, as the canonical extension of P, look like?

• $U \in \mathrm{Fsat}(X_P)$ is a closed element iff there is a filter F of P such that

$$U = {\uparrow}F$$

in $\mathsf{F}(P)$.

• $U \in \mathrm{Fsat}(X_P)$ is an open element iff there exists an ideal H of P such that

$$U = \{G \in \mathsf{F}(P) : G \cap H \neq \emptyset\}.$$

A question

In M. Gehrke, R.J., A. Palmigiano, Δ_1-*completions of a Poset*, Order 1 (2013), different completions of a poset are studied in a systematic way, each one corresponding to a choice of "filters" and "ideals".

The choice of filters and ideals considered in this talk provides the canonical extension, as defined, but other choices provide different completions.

For which choice of "filter" the topological approach just presented can be carried trough?

Thank you!!