New properties of Sasaki projections

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- Our results
- Conclusion

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- The structure of the *lattice of projection operators* $\mathbb{P}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$.
- The lattice of closed subspaces of a separable infinite dimensional Hilbert space.
- *Quantum logic* can be characterised as the logical structure based on orthomodular lattices.

Definition

An *orthocomplementation* on a bounded partially ordered set P is a unary operation with the properties

- x' is the lattice–theoretical complement of x:

$$\begin{aligned} x \wedge x' &= \mathbf{0} \\ x \vee x' &= \mathbf{1} \end{aligned}$$

- $'$ is order reversing: $x \leqslant y \Rightarrow y' \leqslant x'$
- $'$ is an involution: $x'' = x$

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- is *modular* in case $\mathcal{H}$ has a finite dimension,

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- is *orthomodular* in case of an infinite dimension.

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- this relation is reflexive and symmetric.
- transitive if and only if L is a Boolean algebra, in this case any pair of elements commutes.

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Proposition

In any orthomodular lattice, if x commutes with y and with z, then x commutes with y', with $y \vee z$ and with $y \wedge z$, as well as with any (ortho–)lattice polynomial in variables y and z.

Sasaki and his projection

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Let p be an element of the orthomodular lattice L and let φ_p a mapping

$$\begin{array}{rcccl} \varphi_p & : & L & \to & L \\ & & a & \mapsto & p \wedge (p' \vee a) \end{array}$$

then φ_p is called the *Sasaki Projection* of a into $[0, p] \subseteq L$.

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Sasaki projection as binary operation

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We will consider the Sasaki projection as a binary operation (sometimes called the *Sasaki map*)

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The Sasaki projection is neither commutative nor associative, it satisfies

$$\begin{array}{lccccc} \text{idempotence} & x * x & = & x & & \\ \text{neutral element} & 1 * x & = & x * 1 & = & x \\ \text{absorption element} & 0 * x & = & x * 0 & = & 0 \end{array}$$

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Further $x \mathrel{\mathrm{C}} (x * y)$, but not $y \mathrel{\mathrm{C}} (x * y)$

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Computer program (Marek Hyčko)
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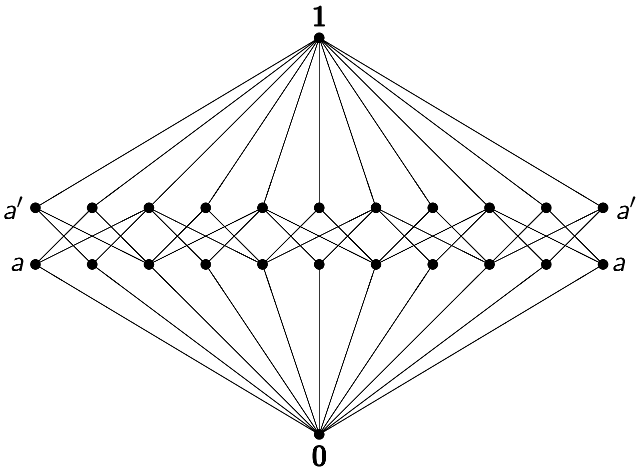
Counterexamples could all be found in the orthomodular lattice L_{22}



Our results

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Lemma

Let L be an orthomodular lattice and let $*$ be the Sasaki projection. It has the following properties:

$$\text{If } x \leqslant y \text{ then } x' \vee (y * z) = x' \vee z.$$

$$\text{If } x \leqslant z \text{ then } x' \vee (y * z) = x' \vee y.$$

$$\text{If } x \leqslant z \text{ then } (x * y) * z = x * (y * z) = x * y.$$

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Note: Neither $x \mathrm{C} z$ nor $y \mathrm{C} z$ are sufficient to fulfil the associativity equation; this is easy to find counterexamples in L_{22}.

Alternative algebras

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Definition

An *alternative algebra* M is a non–empty set together with a binary operation, $*$ which need not to be associative, but it has to be *alternative*, this means that $\forall x, y \in M$ the following equations hold

$$\begin{aligned} x*(x*y) &= (x*x)*y && \text{left identity and} \\ (y*x)*x &= y*(x*x) && \text{right identity} \end{aligned}$$

From the left and right identities the flexible identity follows

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It fulfils also $(x * x') * y = x * (x' * y) = \mathbf{0}$

Moufang–like identities

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Theorem

Let L be an orthomodular lattice and let $*$ be the Sasaki projection. Then

$$\begin{aligned}(x*y*x)*z &= (x*y)*(x*z)\\ \big(z*(x*y)\big)*x &= z*(x*y*x)\\ \big((x*y)*z\big)*x &= (x*y)*(z*x)\end{aligned}$$

for any $x, y, z \in L$.

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- If $x \leqslant z$ or $y \leqslant z$ then

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- Sasaki operations satisfy more equations than most orthomodular operations and therefore they are the best candidates for developing alternative algebraic tools for computations in orthomodular lattices.
- The potential of using Sasaki operations in the algebraic foundations of orthomodular lattices is still not sufficiently exhausted.

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Questions?