A General Extension Theorem for Complete Partial Orders

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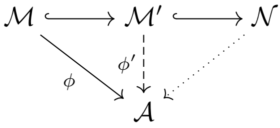
Outline

- Proofs by Means of Zorn's Lemma
- Extension Patterns: Definition and Examples
- GET: A General Extension Theorem
- Applications of GET
- AC $\leftrightarrow$ GET
- Perspectives

Proving Extension Theorems

The Usual Strategy with Zorn's Lemma

- Question: Does a certain morphism ϕ extend "totally"?
- Under suitable conditions, Zorn's Lemma (ZL) postulates the existence of a maximal intermediary extension ϕ'



- If ϕ' were not total, it could be strictly extended by a "one-step" argument. A contradiction to the maximality of ϕ' would arise
- This "one-step" typically involves an actual construction (e.g. Hahn-Banach Theorem, Baer's Criterion)

Extension Patterns

Definition

Let $(E, \leq)$ *be a poset. An* **extension pattern** *for* E *is a triple* $(S, \Vdash, f)$ *where*

- S *is a set*
- $\Vdash \subseteq S \times E$ *is a relation*
- $f : S \times E \to E$ *is a function, such that*

$$\forall x \in S \; \forall e \in E \; \big(\; e \leq f(x, e) \; \wedge \; x \Vdash f(x, e) \; \big).$$

An element $e \in E$ *is called* **total** *if*

$$\forall x \in S \;\; (\; x \Vdash e \;)$$

Note that f captures the "one-step principle"

Extension Patterns

Examples

- *Constant Pattern*

  $(E, \leq)$ poset
  S set
  $\Vdash = S \times E$
  $f(x, e) = e$ for all $x \in S, e \in E$

  Note that every element of E is total for the constant pattern

Extension Patterns

Examples

- *Constant Pattern*

  $(E, \leq)$ poset
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  $f(x, e) = e$ for all $x \in S, e \in E$

  Note that every element of E is total for the constant pattern

- *Maximal Pattern*

  $(E, \leq)$ poset
  $S = E$
  $x \Vdash e$ iff $e \not< x$

  $$f(x, e) = \begin{cases} x & \text{if } e < x \\ e & \text{otherwise} \end{cases}$$

Extension Patterns

Examples

- *Closure Operators*

  S set
  $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ closure operator
  $E \subseteq \mathcal{P}(S)$
  $x \Vdash X$ iff $x \in \mathrm{cl}(X)$

  $$f(x, X) = \begin{cases} X & \text{if } x \in \mathrm{cl}(X) \\ X \cup \{x\} & \text{otherwise} \end{cases}$$

  Example

  S vector space
  $\mathrm{cl}(X) = \mathrm{span}(X)$

General Extension Theorem (GET)

Let E be a cpo with extension pattern
Then over every element of E there is a total one

ZL $\rightarrow$ GET

Maximal elements are total:

Let $e \in E$ be maximal and let $x \in S$
By extension $e \leq f(x, e)$ and $x \Vdash f(x, e)$
Hence $e = f(x, e)$ and thus $x \Vdash e$
Therefore $x \Vdash e$ for all $x \in S$

Note that total elements need not be maximal
(e.g. constant pattern)

Applying GET

- **GET $\to$ ZL**

  Let $e \in E$ be total for the maximal pattern, then

  $$\forall x \in E \ (\ x \nless e \ )$$

  hence e is maximal

Applying GET

- **GET → ZL**

  Let $e \in E$ be total for the maximal pattern, then

  $$\forall x \in E \; (\; x \not< e \;)$$

  hence e is maximal

- **GET → Vectorspace Basis**

  A linearly independent subset $L \subseteq V$ which is total has

  $$\forall v \in V \; (\; v \Vdash L \;), \; \text{i.e. span}(L) = V$$

  hence is a basis

Applying GET

- **GET $\to$ ZL**

  Let $e \in E$ be total for the maximal pattern, then

  $$\forall x \in E \; (\; x \not< e \;)$$

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- **GET $\to$ Vectorspace Basis**

  A linearly independent subset $L \subseteq V$ which is total has

  $$\forall v \in V \; (\; v \Vdash L \;), \text{ i.e. } \mathrm{span}(L) = V$$

  hence is a basis

- GET is equivalent to the Axiom of Choice
  (with respect to a suitable fragment of ZF set theory)

Every strict partial order can be extended to a strict total order

Let $<$ be a strict partial order on a set X

Let E be the cpo of extensions $\prec$ of $<$

Let $S = \{(x, y) \in X \times X : x \neq y\}$

The relation $\Vdash \subseteq S \times E$ is defined by

$$(x, y) \Vdash \prec \quad \text{iff} \quad x \prec y \text{ or } y \prec x$$

Applying GET

Szpilrajn's Extension Theorem

We define $f : S \times E \to E$, extending $\prec$ by (x, y), as follows:

If $(x, y) \nVdash \prec$, then $f\big((x, y), \prec\big) = \prec'$ is defined by

$$a \prec' b \qquad \text{iff} \qquad a \prec b \quad \text{or} \quad a \preceq x \text{ and } y \preceq b$$

If $(x, y) \Vdash \prec$, then $f\big((x, y), \prec\big) = \prec$

If $\prec \in E$ is total for this pattern, then

$$\forall x \forall y \big(x \neq y \;\to\; (x \prec y) \vee (y \prec x)\big)$$

Hence a total element for this pattern is a total order on X which extends $<$

Extension Theorems via GET

An incomprehensive list of examples

- AC
- WO
- ZL
- Vectorspace Bases
- Hahn-Banach Theorem
- Baer's Criterion for Injective Modules
- Sikorski's Theorem for Complete Boolean Algebras
- Szpilrajn's Extension Theorem

Conservativity

- Extension theorems can be considered as conservativity results, e.g. Hahn-Banach, Szpilrajn

"[...] no more may be proved about the subspace A in terms of functionals on the seminormed space B than may already be proved by considering only functionals on the subspace A."

(Mulvey/Pelletier, "A Globalization of the Hahn-Banach Theorem")

- A suitable reformulation of GET might allow to achieve a general conservativity statement. For this we expect to give a direct proof by Open Induction, and thus to gain a constructive proof for every sufficiently concrete instantiation

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