Effective codescent morphisms of distributive lattices

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Looking for a characterization for effectiveness of codescent can be done in two different ways. Accordingly, this talk is organized as follows:

- Topological approach.
- Algebraic Approach.
- Do they give similar results?

1. Topological approach

For every a continuous map of Priestley spaces $p : E \longrightarrow B$, the pullback functor $p^* : \mathcal{PS}p \downarrow B \longrightarrow \mathcal{PS}p \downarrow E$ has a left adjoint $p!$ which is defined by composition with p on the left.

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Let $\mathbb{T}$ be the monad induced by this adjunction and

$$\Phi : \mathcal{PS}p \downarrow B \to (\mathcal{PS}p \downarrow E)^{\mathbb{T}} \cong Des(p)$$

be the Eilenberg-Moore comparison functor.

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A morphism $p : E \longrightarrow B$ is a descent morphism if Φ is full and faithful.

It is an effective descent morphism if Φ is an equivalence of categories.

The finite case: descent in Ord

$\mathcal{F}in\mathcal{P}Sp$ is equivalent to $\mathcal{F}in\mathcal{O}rd$.

Proposition 1.1. For a morphism $p : E \longrightarrow B$ in $\mathcal{O}rd$ (or in $\mathcal{F}in\mathcal{O}rd$) TFAE:

(i) p is a descent morphism;

(ii) for every $b_0 \leq b_1$ in B there exists $e_0 \leq e_1$ in E such that $p(e_i) = b_i$, for i=0,1.

Theorem 1.2. For $p : E \to B$ in $\mathcal{O}rd$ (or in $\mathcal{F}in\mathcal{O}rd$) TFAE:

(a) p is an effective descent morphism;

(b) p is a pullback stable regular epimorphism and, for every pullback in $\mathcal{R}el$

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow{p} & B, \end{array}$$

$A \in \mathcal{O}rd$ whenever $E \times_B A \in \mathcal{O}rd$.

(c) for every $b_0 \leq b_1 \leq b_2$ in B there exists $e_0 \leq e_1 \leq e_2$ in E such that $p(e_i) = b_i$, for i=0,1,2.

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(G.Janelidze and M.S., 2002)

Important tool

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Let $\mathcal{C}$ be a category with pullbacks and coequalizers and $\mathcal{D}$ a full subcategory of $\mathcal{C}$ closed under pullbacks. For a morphism p in $\mathcal{D}$ that is an effective descent morphism in $\mathcal{C}$ the following are equivalent:

(i) p is an effective descent morphism in $\mathcal{D}$;

(ii) for every pullback in $\mathcal{C}$

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow{p} & B \end{array}$$

$E \times_B A \in \mathcal{D} \Rightarrow A \in \mathcal{D}$.

Ordered Stone spaces = Stone spaces equipped with an order relation.

Proposition 1.3. A morphism in $\mathcal{O}rd\mathcal{S}tone$ is a descent morphism if and only if for every $b_0 \leq b_1$ there exists $e_0 \leq e_1$ such that $p(e_i) = b_i$, for i=0,1.

Theorem 1.4. For $p : E \longrightarrow B$ in $\mathcal{O}rd\mathcal{S}tone$ TFAE:

(a) p is an effective descent morphism;

(b) p is a pullback stable regular epimorphism and, for every pullback $\mathcal{R}el\mathcal{S}tone$ (= Stone spaces with a binary relation.)

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$A \in \mathcal{O}rd\mathcal{S}tone$ whenever $E \times_B A \in \mathcal{O}rd\mathcal{S}tone$;

(c) for every $b_0 \leq b_1 \leq b_2$ in B there exists $e_0 \leq e_1 \leq e_2$ in E such that $p(e_i) = b_i$, for i=0,1,2.

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Descent for Priestley spaces

Proposition 1.5. A morphism $\boldsymbol{p : E \longrightarrow B}$ in $\boldsymbol{\mathcal{PS}p}$ is a descent morphism if and only if for every $\boldsymbol{b_0 \leq b_1}$ there exists $\boldsymbol{e_0 \leq e_1}$ such that $\boldsymbol{p(e_i) = b_i}$, for i=0,1.

Theorem 1.6. In the following (a) ⇔ (b) + (c):

(a) p is an effective descent morphism;

(b) p is a pullback stable regular epimorphism and, for every pullback in $\mathcal{O}rd\mathcal{S}tone$ (or in $\mathcal{R}el\mathcal{S}tone$)

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$A \in \mathcal{PS}p$ whenever $E \times_B A \in \mathcal{PS}p$;

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(c) for every $b_0 \leq b_1 \leq b_2$ in B there exists $e_0 \leq e_1 \leq e_2$ in E such that $p(e_i) = b_i$, for i=0,1,2.

(M. Dias and M.S., 2006; G. Janelidze and M.S., 2014)

Does (b) follow from (c)?

This is the case when B is finite, and so we have:

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Theorem 1.7. A morphism $p\colon E \longrightarrow B$, with B finite, is an effective descent morphism if and only if for every $b_0 \leq b_1 \leq b_2$ in B there exists $e_0 \leq e_1 \leq e_2$ in E such that $p(e_i) = b_i$, for i=0,1,2.

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Also true for classes of morphisms that are

(i) surjective;

(ii) open and order-open (or order-closed),

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providing wide classes of effective descent morphisms in $\mathcal{PS}p$. (G.Janelidze and M.S., 2014)

An open problem

- Describe morphisms $p: E \longrightarrow B$ in $\mathcal{PS}p$ such that, for every morphism $\alpha: A \longrightarrow B$ in $\mathcal{O}rd\mathcal{S}tone$ (or $\mathcal{R}el\mathcal{S}tone$), A belongs to $\mathcal{PS}p$ whenever $E \times_B A$ belongs to $\mathcal{PS}p$.

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This holds for morphisms p lifting three chains $b_0 \leq b_1 \leq b_2$, that is satisfying condition (c), in the finite case and also when just B is finite.

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This holds for morphisms p lifting three chains $b_0 \leq b_1 \leq b_2$, that is satisfying condition (c), in the finite case and also when just B is finite.

Such morphisms necessarily lift three chains but this does not seem to be a sufficient condition, in general.

Proposition 2.1. Let $p\colon B \longrightarrow E$ be a homomorphism of distributive lattices. Then p is a codescent morphism, that is a pushout stable regular monomorphism, if and only if for for every pair (b_0, b_1) of prime ideals of B with $b_0 \subseteq b_1$ there exist prime ideals $e_0 \subseteq e_1$ in E with $p^{-1}(e_i) = b_i, i = 0, 1$.

Theorem 2.2. Let $p\colon B \to E$ be a homomorphism of distributive lattices with finite B. Then p is an effective codescent morphism, that is it makes the induced pushout functor

$$B \downarrow \mathcal{DL}at \to E \downarrow \mathcal{DL}at$$

comonadic, if and only if for for every triple (b_0, b_1, b_2) of prime ideals of B with $b_0 \subseteq b_1 \subseteq b_2$ there exist prime ideals $e_0 \subseteq e_1 \subseteq e_2$ in E with $p^{-1}(e_i) = b_i, i = 0, 1, 2$.

3. Algebraic approach

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Theorem 3.1. Let $p : B \longrightarrow E$ be a homomorphism of commutative monoids in a monoidal category $\mathcal{X}$ for which there exists a morphism $q : E \longrightarrow B$ in the category $\mathcal{X}^B$, of B-actions, with $q \cdot p = 1_B$. Then p is an effective descent morphism of commutative monoids.

Commutative monoids in semilattices

Then taking $\mathcal{X}$ the category of semilattices, and since a subsemiring of a semiring of idempotents consists itself of idempotents, we conclude:

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Corollary 3.2. Let $p : B \longrightarrow E$ be a homomorphism of distributive lattices, for which there exists a map $q : E \longrightarrow B$ with $q(0) = 0$, $q(e \vee e') = q(e) \vee q(e')$,

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such that $q \cdot p = 1_B$.

Then p is an effective codescent morphism of distributive lattices.

4. Do they give similar results?

The answer is far from being “ yes". However in the finite case:

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- Theorem 1.2 tells us that being an effective descent map is much weaker than being open and even much weaker than being a triquotient.

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- Theorem 1.2 tells us that being an effective descent map is much weaker than being open and even much weaker than being a triquotient.

Triquotients of finite topological spaces are exactly those maps $p : E \longrightarrow B$ such that for every $b_0 \le b_1 \le \cdots \le b_n$ in B there exists $e_0 \le e_1 \le \cdots \le e_n$ in E with $p(e_i) = b_i$, for every natural number n. (The necessity was proved by G. Janelidze and M.S., 2002 and the sufficiency by M.M.Clementino, 2002)

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- Theorem 1.2 tells us that being an effective descent map is much weaker than being open and even much weaker than being a triquotient.

- Corollary 3.2 gives a sufficient condition that is *between* “open" and “triquotient", comparing with the notions of open and triquotient for locales which are monoids in the monoidal category of complete semilattices.

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