Ramsey's Theorem for pairs in k colors In the Hierarchy of Sub-Classical Principles For Intuitionistic Arithmetic

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Introducing Ramsey Theorem

- Ramsey Theorem solves questions like: if you have ω people at a party, is there some **infinite** subset whose members all know each other or an **infinite** subset none of whose members know each other?
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- Below an example with a color assignment for the nodes: $\{0, 1, 2, 3, 4, 5\}$. 0 knows $2, 3, 5$ and does not know $1, 4$.

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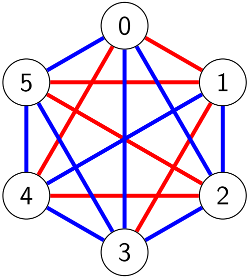

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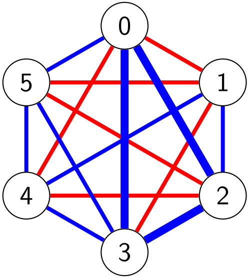

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1-colorings of a graph

- **Ramsey Theorem** says that infinite complete graphs with edges in finitely many colors have infinite homogeneous subsets.
- An intermediate step in the quest for an homogeneous (all-red, all-blue, . . .) subset X is a subset Y with some 1-**coloring**.
- A subset Y has some 1-coloring if all edges from **the same point of** Y to points of Y having a larger index have the **same color**.
- This color is called the **color in Y of the point**. Below a 1-colored set Y.

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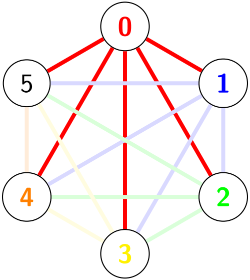

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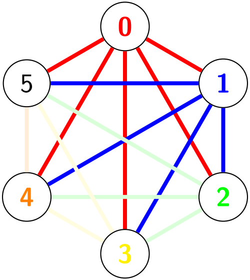

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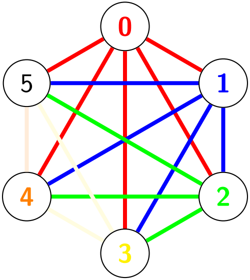

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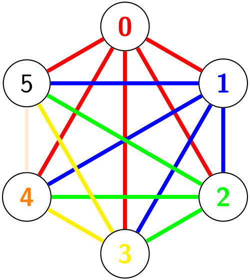

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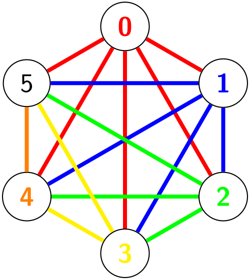

RT^2_2: Ramsey Theorem in 2 colors for pairs

- In a formal statement of Ramsey we consider pairs $\{i, j\}$ of **natural numbers** with $i \neq j$.
- It is not restrictive to assume that $i < j$: we represent all pairs exactly once.
- X is an **homogeneous** set w.r.t. the color assignment C and the color c if all $i < j$ in X have color c.
- RT^2_2 **(Ramsey for pairs and for** 2 **colors) says**: if you have an assignment C of two colors $1, 2$ to all pairs $i < j$ of natural numbers, then there is some color $c = 1, 2$ and some **infinite** set X of natural numbers which is homogeneous w.r.t. C and the color c.

Theorem (RT^2_n: Ramsey Theorem in n colors for graphs)

If you have an assignment of n colors $1, \ldots, n$ to all pairs of different natural numbers, then there is some color $c = 1, \ldots, n$ and some infinite set X homogeneous w.r.t. C and the color c.

Let PA be First Order Classical Arithmetic.

- If the coloring C is defined in the **in the language of PA** (for instance, C is recursive), then RT^2_n may be proved **in PA**, that is: we may define some set X by some arithmetical predicate then prove that X is homogeneous [2].
- Ramsey Theorem is **not effective**. There is no recursive map:
  1. taking some recursive coloring C in input
  2. providing as output some color c and an arithmetical formula describing some homogeneous set X for C and c ([2]).
- As a corollary, RT^2_2 and RT^2_n are purely classical results: they **cannot be proved in HA**, Intuitionistic First Order Arithmetic.

Theorem (RT^2_n: Ramsey Theorem in *n* colors for graphs)

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How far is Ramsey's Theorem from being constructive?

A non-intuitionistic theorem may still have some constructive feature. Let HA be Intuitionistic Arithmetic. These results are taken from [4].

- If we use HA + **Markov principle** to derive $\exists x.P(x, a)$, then the proof contains a method recursive in a to find x, but no estimate of the number of steps which are required.
- If we use HA + Σ^0_1-LLPO **(König's Lemma for recursive trees)** to derive $\exists x.P(x, a)$, then the proof contains a method recursive in a to find some finite set I such that $x \in I$.
- If we use HA + EM-1 **(Excluded Middle for semi-decidable formulas)** to derive to derive $\exists x.P(x, a)$, then the proof contains some limit computable map $f(a)$ such that $x = f(a)$.

Unfortunately, for RT^2_2 we have a negative result: RT^2_2 corresponds to $\mathbf{\Sigma^0_3}$-LLPO **(König's Lemma for $\mathbf{\Sigma^0_2}$-trees)**, a sub-classical principle **quite high** in the hierarchy of sub-classical principles.

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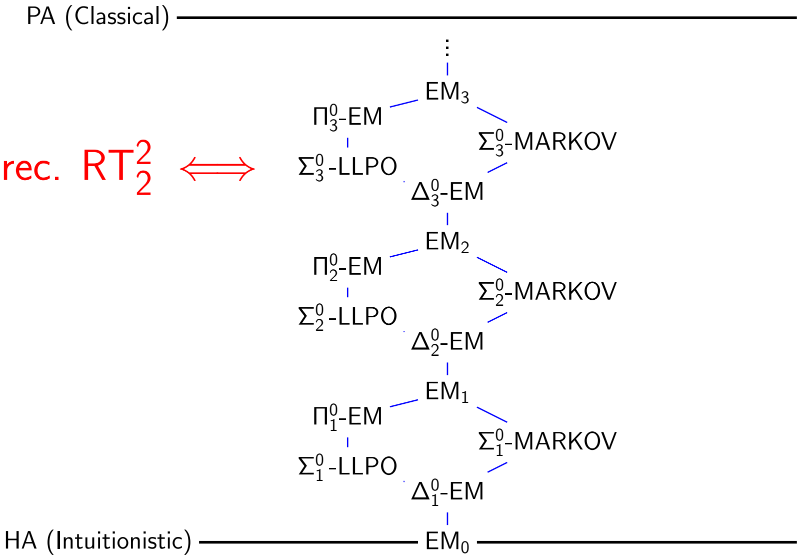
RT^2_2 is a sub-classical principle

We proved in [5] that RT^2_2 for **recursive coloring** is König's Lemma for Σ^0_2-trees, in the hierarchy of formulas provable in HA introduced in [4].



RT^2_2 and RT^2_n for recursive coloring are the same sub-classical principle

1. The goal of this talk is to prove:

   $$(\mathrm{RT}^2_n \textbf{ for rec. col.}) \iff (\mathrm{RT}^2_2 \textbf{ for rec. col.}) \iff \boldsymbol{\Sigma^0_3}\text{-LLPO}.$$

2. We **cannot use the obvious proof** of $\mathrm{RT}^2_2 \implies \mathrm{RT}^2_n$ by induction over the number of colors, because this proof requires **non-recursive colorings**.
3. We already have intuitionistic proofs of (RT^2_n for rec. coloring) $\implies$ (RT^2_2 for rec. coloring) (immediate) and of (RT^2_2 for rec. coloring) $\implies$ Σ^0_3-LLPO ([5]).
4. **All we need** is some intuitionistic proof of Σ^0_3-LLPO $\implies$ (RT^2_n for rec. coloring).
5. To this aim, we translate **Jockusch's proof** [2] of RT^2_n in HA + Σ^0_3-LLPO,
6. We modify Jockusch's proof whenever this proof uses a **sub-classical principle stronger than** Σ^0_3-LLPO

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Jockusch's proof of RT^2_n in PA

- Fix a coloring c of all pairs $i < j$ of natural numbers.
- We say that a set Y of numbers has a 1-coloring if for all $i < j < k$ the coloring of $\{i, j\}$ and $\{i, k\}$ is the same. The common color of all edges from i is called **the** 1-**color of the number** i **in** Y.
- Jockusch's proof in PA is based on the existence of an infinite set Y with 1-coloring.
- If we have an infinite set Y with some 1-coloring, then by the infinite Pigeonhole Principle there is some infinite set $X \subseteq Y$ of numbers **all of the same** 1-**color**: that is, all edges from any numbers in X to any number in X have the same color.
- By definition, X is homogeneous: RT^2_n follows.
- Jockusch's proof cannot be carried over in $\mathrm{HA} + \Sigma^0_3$-LLPO: we now explain why.

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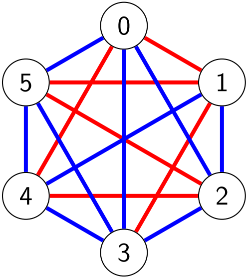
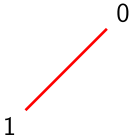
Erdős' trees

- From each assignment C of n colors to all pairs of natural numers, Jockusch defines an infinite n-ary tree T, including all natural numbers, and whose branches are 1-coloring.
- These tree are called **Erdős' trees**.
- **Example** We assume be given a coloring c on $\{0, 1, 2, 3, 4, 5\}$ and we define some corresponding Erdős' tree T. All branches of T are some 1-coloring.





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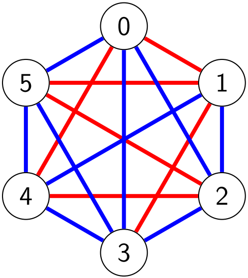

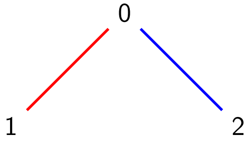

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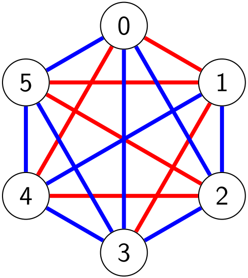

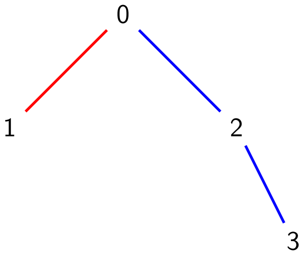

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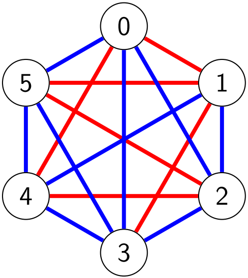

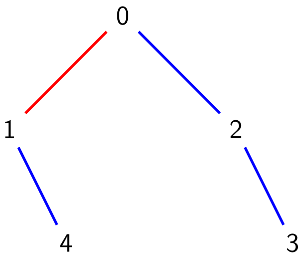

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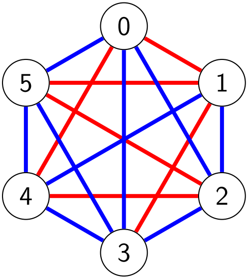

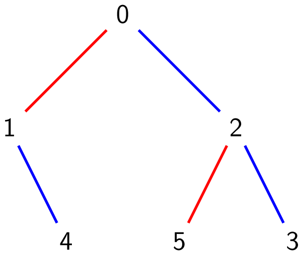

Jockusch's proof of RT^2_n in HA

- From each **recursive** assignment C of n colors to all pairs of natural numers, Jockusch defines an infinite Π^0_1 Erdős' tree: an n-ary tree T, including all natural numbers, and whose branches are 1-coloring.
- Jockusch deduces, using König's Lemma, that T has some infinite Π^0_2-branch Y.
- By definition of Erdős' tree, the branch Y is an infinite 1-coloring.
- Jockusch concludes, using the Infinite Pigeonhole Principle for the Π^0_2-branch Y, that there is some infinite set of numbers $X \subseteq Y$, with all numbers of the same 1-color. X is the homogeneous set required.
- Jockusch's proof cannot be carried in HA out using Σ^0_3-LLPO, because Σ^0_3-LLPO does **not** imply the Infinite Pigeonhole Principle for the Π^0_2-sets.

A proof in HA of: Σ^0_3-LLPO $\implies$ (RT^2_n for recursive coloring)

Our contribution is to define, from any **recursive** color assignment C on pairs of natural numbers, some particular infinite Erdős' tree T, with one extra property:

any 1-*color occurring* **infinitely many times** *in* T
occurs **infinitely many times** *in any infinite branch* Y

- Thus, in order to prove there is a 1-color occurring infinitely many times in Y, we prove there is a color of numbers occurring infinitely many times in T.
- This version of Jockusch's proof may be carried in HA out using Σ^0_3-LLPO
- Indeed, Σ^0_3-LLPO implies the Infinite Pigeonhole Principle for the Π^0_1-sets, and T is a Π^0_1-set (while Y is not).

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