Subordinations, closed relations and compact Hausdorff spaces

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Joint work with Guram Bezhanishvili, Nick Bezhanishvili, and Yde Venema.

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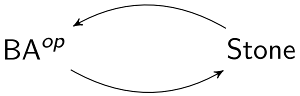
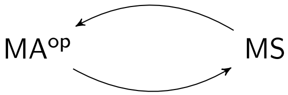
Stone Duality



Stone duality (1936)

A Stone space is a compact Hausdorff and zero-dimensional space.

Jónsson-Tarski Duality



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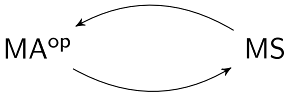

Jónsson-Tarski duality

A modal algebra $(B, \Diamond)$ is a pair where B is a Boolean algebra and $\Diamond$ is a unary operation which satisfies:

(i)$\Diamond 0 = 0$
(ii) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$.

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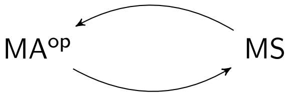

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A modal space is a Stone space X with a relation R which satisfies:

(i) $R[x]$ is a closed set for each $x \in X$
(ii) $R^{-1}(U)$ is a clopen set for each clopen $U \subseteq X$.

Jónsson-Tarski Duality

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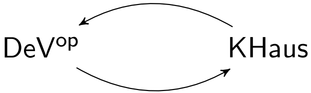
This leads to a dual equivalence between the category MS of modal spaces and continuous p-morphisms ($f \circ R = R \circ f$), and the category MA of modal algebras and their homomorphisms.

de Vries duality for Compact Hausdorff Spaces



De Vries duality for KHaus

de Vries algebras [de Vries (1962)]

A de Vries algebra is a pair $(A, \prec)$ consisting of a complete Boolean algebra A and a binary relation $\prec$ on A satisfying the following

(S1) $0 \prec 0$ and $1 \prec 1$;

(S2) $a \prec b, c$ implies $a \prec b \wedge c$;

(S3) $a, b \prec c$ implies $a \vee b \prec c$;

(S4) $a \leq b \prec c \leq d$ implies $a \prec d$.

(S5) $a \prec b$ implies $a \leq b$;

(S6) $a \prec b$ implies $\neg b \prec \neg a$;

(S7) $a \prec b$ implies there is $c \in B$ with $a \prec c \prec b$;

(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

de Vries algebras

The set of regular open sets ($U = \mathbf{IC}U$) of a compact Hausdorff space X form a complete Boolean algebra.

For $U, V \in \mathrm{RO}(X)$ define $U \prec V$ if $\mathbf{C}U \subseteq V$. Then $(\mathrm{RO}(X), \prec)$ is a de Vries algebra.

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Each de Vries algebra is isomorphic to a de Vries algebra of the form $(\mathrm{RO}(X), \prec)$ for some compact Hausdorff space X.

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The goal of this talk is to obtain a "modal"-like duality for de Vries algebras.

Subordinations

Definition

A subordination on a Boolean algebra B is a binary relation $\prec$ satisfying:

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(S2) $a \prec b, c$ implies $a \prec b \wedge c$;

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Examples of subordinations (that satisfy additional conditions) are modal operators $\Box$ and de Vries compingent relations.

Alternatively subordinations can be described by pre-contact relations (Düntsch and Vakarelov) and quasi-modal operators (Celani).

Closed relations

Subordinations can be dually described by means of closed relations.

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Lemma

Let X be a compact Hausdorff space and let R be a binary relation on X. The following conditions are equivalent.

1. R is a closed relation.
2. For each closed subset F of X, both $R[F]$ and $R^{-1}[F]$ are closed.
3. If $(x, y) \notin R$, then there is an open neighborhood U of x and an open neighborhood V of y such that $R[U] \cap V = \varnothing$.

Duality for Subordinations

Let Sub be the category whose objects are pairs $(B, \prec)$, where B is a BA and $\prec$ is a subordination on B, and whose morphisms are Boolean homomorphisms h satisfying $a \prec b$ implies $h(a) \prec h(b)$.

Let StR be the category whose objects are pairs (X, R), where X is a Stone space and R is a closed relation on X, and whose morphisms are continuous stable morphisms[1].

[1] We say $f : X_1 \to X_2$ is stable if xR_1y implies $f(x)R_2f(y)$.

Duality for Subordinations

For $(B, \prec) \in \mathsf{Sub}$, let $(B, \prec)_* = (X, R)$, where X is the Stone space of B and xRy iff $\twoheaduparrow x \subseteq y$, where $\twoheaduparrow x = \{b \in B : \exists a \in x \text{ such that } a \prec b\}$. Then $(X, R) \in \mathsf{StR}$

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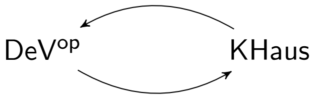
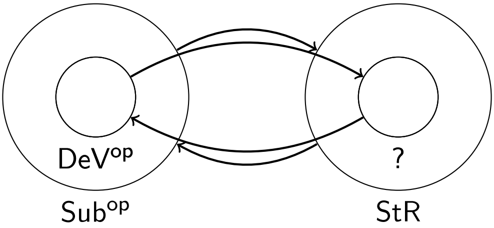
Theorem

The categories Sub *and* StR *are dually equivalent.*

A "modal" de Vries duality?





Elementary conditions

Let $(B, \prec)$ be a subordination, which satisfies the following axioms.

(S5) $a \prec b$ implies $a \leq b$;

(S6) $a \prec b$ implies $\neg b \prec \neg a$;

(S7) $a \prec b$ implies there is $c \in B$ with $a \prec c \prec b$;

Lemma

Let $(X, R) \in \mathrm{StR}$ be the dual space of $(B, \prec)$.

1. *R is reflexive iff $\prec$ satisfies (S5).*
2. *R is symmetric iff $\prec$ satisfies (S6).*
3. *R is transitive iff $\prec$ satisfies (S7).*

Irreducible equivalence relations

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We call a closed equivalence relation R on a compact Hausdorff space X irreducible if the factor-map $\pi : X \to X/R$ is irreducible.

A closed equivalence relation R is irreducible iff for each proper closed subset F of X, we have $R[F]$ is a proper subset of X.

Gleason spaces

(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

Lemma

Let $(B, \prec)$ satisfy (S1-S7), and let (X, R) be the dual of $(B, \prec)$. Then the closed equivalence relation R is irreducible iff $\prec$ satisfies (S8).

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We call a pair (X, R) a Gleason space if X is an extremally disconnected space (each regular open is clopen) and R is an irreducible equivalence relation on X.

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Theorem

Gle *is dually equivalent to* DeV, *hence* Gle *is equivalent to* KHaus.

Future work

For details see:

Subordinations, closed relations and compact Hausdorff spaces. Guram Bezhanishvili, Nick Bezhanishvili, Sumit Sourabh, Yde Venema, available at `http://www.illc.uva.nl/Research/Publications/Reports/PP-2014-23.text.pdf`, December 2014.

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- Develop a finitary calculus for compact Hausdorff spaces.
- Characterize the class of axioms on a subordination which corresponds to elementary conditions on the dual Stone space.

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Thank you!