Using topological systems to create a framework for institutions

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Outline

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2. Affine systems
3. Affine theories
4. Affine institutions
5. Conclusion
Institutions

- There is a convenient approach to logical systems in computer science based in *institutions* of J. A. Goguen and R. M. Burstall.
- An institution comprises a category of (abstract) signatures, where every signature has its associated sentences, models, and a relation of satisfaction, which is invariant under change of signature, i.e., “truth is invariant under change of notation”.
- Institutions include unsorted universal algebra, many-sorted algebra, order-sorted algebra, first-order logic, partial algebra.
- A number of authors proposed generalizations of institutions in various forms, e.g., using a purely category-theoretic approach.
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Topological systems

- Based in the ideas of geometric logic, *topological systems* of S. Vickers provide a common setting for both topological spaces and their underlying algebraic structures—locales.

- S. Vickers showed system spatialization and localication procedures, i.e., ways to move back and forth between the categories of topological spaces (resp., locales) and topological systems.

- Recently, topological systems have gained in interest in connection with lattice-valued topology, e.g., one has
  - introduced and studied lattice-valued topological systems;
  - provided lattice-valued system spatialization procedure.
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Institutions versus systems

To find relationships between institutions and topological systems, J. T. Denniston, A. Melton, and S. E. Rodabaugh introduced *lattice-valued institutions*, and showed that lattice-valued topological systems provide their particular instance.

A. Sernadas, C. Sernadas, and J. M. Valença introduced crisp *topological institutions* based in topological systems, the slogan being that “the central concept is the theory, not the formula”.

The purpose of this talk is to show that a suitably generalized concept of topological system provides a setting for *elementary institutions* of A. Sernadas, C. Sernadas, and J. M. Valença.
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Definition 1

Let \( \Omega = (n_{\lambda})_{\lambda \in \Lambda} \) be a family of cardinal numbers, which is indexed by a (possibly proper or empty) class \( \Lambda \).

- An **\( \Omega \)-algebra** is a pair \( (A, (\omega^A_{\lambda})_{\lambda \in \Lambda}) \), comprising a set \( A \) and a family of maps \( A^{n_{\lambda}} \xrightarrow{\omega^A_{\lambda}} A \) (\( n_{\lambda} \)-ary primitive operations on \( A \)).

- An **\( \Omega \)-homomorphism** \( (A_1, (\omega^{A_1}_{\lambda})_{\lambda \in \Lambda}) \xrightarrow{\varphi} (A_2, (\omega^{A_2}_{\lambda})_{\lambda \in \Lambda}) \) is a map \( A_1 \xrightarrow{\varphi} A_2 \) such that \( \varphi \circ \omega^{A_1}_{\lambda} = \omega^{A_2}_{\lambda} \circ \varphi^{n_{\lambda}} \) for every \( \lambda \in \Lambda \).

**\( \text{Alg}(\Omega) \)** is the construct of \( \Omega \)-algebras and \( \Omega \)-homomorphisms.
Definition 2

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of $\Omega$-homo-morphisms with injective (resp. surjective) underlying maps. A variety of $\Omega$-algebras is a full subcategory of $\text{Alg}(\Omega)$, which is closed under the formation of products, $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients, and whose objects (resp. morphisms) are called algebras (resp. homomorphisms).
Algebraic preliminaries

Examples of varieties

Example 3

1. \( \text{CSLat}(\lor) \) is the variety of \( \lor \)-semilattices, and \( \text{CSLat}(\land) \) is the variety of \( \land \)-semilattices.

2. \( \text{Frm} \) is the variety of frames.

3. \( \text{CBA} \text{Alg} \) is the variety of complete Boolean algebras.

4. \( \text{CSL} \) is the variety of closure semilattices, i.e., \( \land \)-semilattices, with the singled out bottom element.
The following extends the notion of *affine set* of Y. Diers.

**Definition 4**

Given a functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{\text{op}}$, where $\mathbf{B}$ is a variety of algebras, $\text{AfSpc}(T)$ is the concrete category over $\mathbf{X}$, whose objects ($T$-affine spaces or $T$-spaces) are pairs $(X, \tau)$, where $X$ is an $\mathbf{X}$-object and $\tau$ is a subalgebra of $TX$; morphisms ($T$-affine morphisms or $T$-morphisms) $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ are $\mathbf{X}$-morphisms $X_1 \xrightarrow{f} X_2$ with the property that $(Tf)^{\text{op}}(\alpha) \in \tau_1$ for every $\alpha \in \tau_2$.

The concrete category $(\text{AfSpc}(T), \dashv | |)$ is topological over $\mathbf{X}$.
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*objects* (\( T \)-affine spaces or \( T \)-spaces) are pairs \((X, \tau)\), where \( X \) is an \( X \)-object and \( \tau \) is a subalgebra of \( TX \);

*morphisms* (\( T \)-affine morphisms or \( T \)-morphisms) \((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)\) are \( X \)-morphisms \( X_1 \xrightarrow{f} X_2 \) with the property that \((Tf)^{\text{op}}(\alpha) \in \tau_1\) for every \( \alpha \in \tau_2 \).

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Examples

Example 5

Given a variety $\mathbf{B}$, every subcategory $\mathbf{S}$ of $\mathbf{B}^{op}$ induces a functor $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{P}_S} \mathbf{B}^{op}$, $\mathcal{P}_S((X_1, B_1) \xrightarrow{(f, \varphi)} (X_2, B_2)) = B_1^{X_1} \xrightarrow{\mathcal{P}_S(f, \varphi)} B_2^{X_2}$, where $(\mathcal{P}_S(f, \varphi))^{op}(\alpha) = \varphi^{op} \circ \alpha \circ f$.

Example 6

1. If $\mathbf{B} = \text{Frm}$, then $\text{AfSpc}(\mathcal{P}_2) = \text{Top}$ (topological spaces).
2. If $\mathbf{B} = \text{CSL}$, then $\text{AfSpc}(\mathcal{P}_2) = \text{Cls}$ (closure spaces).
3. $\text{AfSpc}(\mathcal{P}_B)$ is the category $\text{AfSet}(B)$ of affine sets of Y. Diers.
4. If $\mathbf{B} = \text{Frm}$, then $\text{AfSpc}(\mathcal{P}_S) = S\text{-Top}$ (variable-basis lattice-valued topological spaces of S. E. Rodabaugh).
Affine spaces

Examples

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Given a variety \( B \), every subcategory \( S \) of \( B^{op} \) induces a functor

\[
\text{Set} \times S \xrightarrow{\mathcal{P}_S} B^{op}, \quad \mathcal{P}_S((X_1, B_1) \xrightarrow{(f, \varphi)} (X_2, B_2)) = B_1^{X_1} \xrightarrow{\mathcal{P}_S(f, \varphi)} B_2^{X_2},
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3. \( \text{AfSpc}(\mathcal{P}_B) \) is the category \( \text{AfSet}(B) \) of affine sets of Y. Diers.
4. If \( B = \text{Frm} \), then \( \text{AfSpc}(\mathcal{P}_S) = \text{S-Top} \) (variable-basis lattice-valued topological spaces of S. E. Rodabaugh).
**Affine systems**

**Definition 7**

Given a functor $X \xrightarrow{T} B^{op}$, $\text{AfSys}(T)$ is the comma category $(T \downarrow 1_{B^{op}})$, concrete over the product category $X \times B^{op}$, whose objects ($T$-affine systems or $T$-systems) are triples $(X, \kappa, B)$, made by $B^{op}$-morphisms $TX \xrightarrow{\kappa} B$; morphisms ($T$-affine morphisms or $T$-morphisms) $(X_1, \kappa_1, B_1) \xrightarrow{(f, \varphi)} (X_2, \kappa_2, B_2)$ are $X \times B^{op}$-morphisms $(X_1, B_1) \xrightarrow{(f, \varphi)} (X_2, B_2)$, making the next diagram commute:

\[
\begin{array}{ccc}
TX_1 & \xrightarrow{Tf} & TX_2 \\
\kappa_1 \downarrow & & \kappa_2 \downarrow \\
B_1 & \xrightarrow{\varphi} & B_2.
\end{array}
\]
### Example 8

1. If $B = \text{Frm}$, then $\text{AfSys}(\mathcal{P}_2) = \text{TopSys}$ (topological systems of S. Vickers).

2. If $B = \text{Set}$, then $\text{AfSys}(\mathcal{P}_B) = \text{Chu}_B$ (Chu spaces over a set $B$ of P.-H. Chu).

### Definition 9

A $T$-system $(X, \kappa, B)$ is called *separated* provided that $TX \xrightarrow{\kappa} B$ is an epimorphism in $B^{op}$. $\text{AfSys}_s(T)$ is the full subcategory of $\text{AfSys}(T)$ of separated $T$-systems.

### Example 10

For $B = \text{CSL}$, $\text{AfSys}_s(\mathcal{P}_2) = \text{SP}$ (state property systems of D. Aerts).
Affine systems

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Example 10

For $B = \text{CSL}$, $\text{AfSys}_s(\mathcal{P}_2) = \text{SP}$ (state property systems of D. Aerts).
Theorem 11

1. \( \text{AfSpc}(T) \xleftarrow{E} \text{AfSys}(T) \), \( E((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1, e^{\tau_1}_{\tau_1}, \tau_1) \xrightarrow{(f, \varphi)} (X_2, e^{\tau_1}_{\tau_2}, \tau_2) \) is a full embedding, with \( e_{\tau_i} \) the inclusion \( \tau_i \hookrightarrow TX_i \) and \( \varphi^{\tau_1}_{\tau_2} \) the restriction \( \tau_2 \xrightarrow{(Tf)^{\tau_1}_{\tau_2}} \tau_1 \).

2. \( E \) has a right-adjoint-left-inverse \( \text{AfSys}(T) \xrightarrow{\text{Spat}} \text{AfSpc}(T) \), \( \text{Spat}((X_1, \kappa_1, B_1) \xrightarrow{(f, \varphi)} (X_2, \kappa_2, B_2)) = (X_1, \kappa_1^{\text{op}}(B_1)) \xrightarrow{f} (X_2, \kappa_2^{\text{op}}(B_2)) \).

3. \( \text{AfSpc}(T) \) is isomorphic to a full (regular mono)-coreflective subcategory of \( \text{AfSys}(T) \).
Theorem 12

\[ E \quad \text{and} \quad \text{Spat} \quad \text{restrict to} \quad \text{AfSpc}(T) \xleftarrow{\text{E}} \text{AfSys}_s(T) \quad \text{and} \quad \text{AfSys}_s(T) \xrightarrow{\text{Spat}} \text{AfSpc}(T), \text{providing an equivalence between the categories} \ \text{AfSpc}(T) \quad \text{and} \quad \text{AfSys}_s(T) \quad \text{such that} \quad \text{Spat} \ E = 1_{\text{AfSpc}(T)}. \]

Corollary 13

\[ \text{AfSpc}(T) \quad \text{is the amnestic modification of} \quad \text{AfSys}_s(T). \]

Example 14

1. \text{Top} \quad \text{is isomorphic to a full (regular mono)-coreflective subcategory of} \quad \text{TopSys (system spatialization procedure of S. Vickers).}
2. \text{The categories CIs and SP are equivalent.}
Consequences

Theorem 12

\[ E \text{ and } \text{Spat} \text{ restrict to } \text{AfSpc}(T) \xleftarrow{E} \text{AfSys}_s(T) \text{ and } \text{AfSys}_s(T) \xrightarrow{\text{Spat}} \text{AfSpc}(T), \text{ providing an equivalence between the categories } \text{AfSpc}(T) \text{ and } \text{AfSys}_s(T) \text{ such that } \text{Spat} \overline{E} = 1_{\text{AfSpc}(T)}. \]

Corollary 13

\[ \text{AfSpc}(T) \text{ is the amnestic modification of } \text{AfSys}_s(T). \]

Example 14

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Theorem 12

\[ E \text{ and } \text{Spat} \text{ restrict to } \text{AfSpc}(T) \xleftarrow{E} \text{AfSys}_s(T) \text{ and } \text{AfSys}_s(T) \xrightarrow{\text{Spat}} \text{AfSpc}(T), \text{ providing an equivalence between the categories } \text{AfSpc}(T) \text{ and } \text{AfSys}_s(T) \text{ such that } \text{Spat} \bar{E} = 1_{\text{AfSpc}(T)}. \]

Corollary 13

\text{AfSpc}(T) \text{ is the amnestic modification of } \text{AfSys}_s(T).

Example 14

1. \text{Top} \text{ is isomorphic to a full (regular mono)-coreflective subcategory of } \text{TopSys} \text{ (system spatialization procedure of S. Vickers).}
2. The categories \text{Cls} \text{ and } \text{SP} \text{ are equivalent.}
**Proposition 15**

\[
AfSys(T) \xrightarrow{\text{Loc}} B^{\text{op}}, \quad \text{Loc}((X_1, \kappa_1, B_1) \xrightarrow{(f, \varphi)} (X_2, \kappa_2, B_2)) = B_1 \xrightarrow{\varphi} B_2
\]

*B_2* is a functor.

**Theorem 16**

Given a functor \(X \xrightarrow{T} B^{\text{op}}\), the following are equivalent.

1. There exists an adjoint situation \((\eta, \varepsilon) : T \dashv \text{Pt} : B^{\text{op}} \to X\).
2. There exists a full embedding \(B^{\text{op}} \xhookrightarrow{E} \text{AfSys}(T)\) such that \(\text{Loc}\) is a left-adjoint-left-inverse to \(E\). \(B^{\text{op}}\) is then isomorphic to a full reflective subcategory of \(\text{AfSys}(T)\).
Proposition 15

\[ \text{AfSys}(T) \xrightarrow{\text{Loc}} \mathcal{B}^{\text{op}}, \text{Loc}((X_1, \kappa_1, B_1) \xrightarrow{(f, \varphi)} (X_2, \kappa_2, B_2)) = B_1 \xrightarrow{\varphi} B_2 \]


B_2 \text{ is a functor.}

Theorem 16

Given a functor \( X \xrightarrow{T} \mathcal{B}^{\text{op}} \), the following are equivalent.

1. There exists an adjoint situation \((\eta, \varepsilon) : T \dashv \text{Pt} : \mathcal{B}^{\text{op}} \rightarrow X\).

2. There exists a full embedding \( \mathcal{B}^{\text{op}} \xhookrightarrow{E} \text{AfSys}(T) \) such that \( \text{Loc} \) is a left-adjoint-left-inverse to \( E \). \( \mathcal{B}^{\text{op}} \) is then isomorphic to a full reflective subcategory of \( \text{AfSys}(T) \).
Remark 17

Every functor $\textbf{Set} \xrightarrow{\mathcal{P}_B} \mathcal{B}^{\text{op}}$ has a right adjoint $\mathcal{B}^{\text{op}} \xrightarrow{\text{Pt}_B} \textbf{Set}$, $\text{Pt}_B(B_1 \xrightarrow{\varphi} B_2) = \mathcal{B}(B_1, B) \xrightarrow{\text{Pt}_B \varphi} \mathcal{B}(B_2, B)$, $(\text{Pt}_B \varphi)(p) = p \circ \varphi^{\text{op}}$.

Example 18

- $\text{Loc}$ is isomorphic to a full reflective subcategory of $\text{TopSys}$, which gives the system localisation procedure of S. Vickers.
- $\mathcal{B}^{\text{op}}$ is isomorphic to a full reflective subcategory of $\text{AfSys}(\mathcal{P}_B)$. 
**Remark 17**

Every functor \( \textbf{Set} \xrightarrow{\mathcal{P}_B} \textbf{B}^{\text{op}} \) has a right adjoint \( \textbf{B}^{\text{op}} \xrightarrow{\text{Pt}_B} \textbf{Set} \),

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\text{Pt}_B(B_1 \xrightarrow{\varphi} B_2) = \textbf{B}(B_1, B) \xrightarrow{\text{Pt}_B \varphi} \textbf{B}(B_2, B), \quad (\text{Pt}_B \varphi)(p) = p \circ \varphi^{\text{op}}.
\]

**Example 18**

- \( \textbf{Loc} \) is isomorphic to a full reflective subcategory of \( \textbf{TopSys} \), which gives the system localisation procedure of S. Vickers.
- \( \textbf{B}^{\text{op}} \) is isomorphic to a full reflective subcategory of \( \textbf{AfSys}(\mathcal{P}_B) \).
One could like to study the properties of the categories $\text{AfSys}(T)$ and $\text{AfSpc}(T)$ through the properties of the functor $X^T \to B^{op}$.

**Definition 19**

An *affine theory* is a functor $X^T \to B^{op}$ with $B$ a variety of algebras.
Affine theories

One could like to study the properties of the categories $\text{AfSys}(T)$ and $\text{AfSpc}(T)$ through the properties of the functor $X \xrightarrow{T} B^{\text{op}}$.

**Definition 19**

An *affine theory* is a functor $X \xrightarrow{T} B^{\text{op}}$ with $B$ a variety of algebras.
Affine theories

**Definition 20**

\textbf{AfTh} is the category given by the following data:

- **objects** are affine theories \( X \xrightarrow{T} B^{\text{op}}; \)

- **morphisms** \( T_1 \xrightarrow{(F, \Phi, \eta)} T_2 \) (shortened to \( \eta \)) comprise two functors \( X_1 \xrightarrow{F} X_2, \ B_1 \xrightarrow{\Phi} B_2 \) and a natural transformation \( T_2 F \xrightarrow{\eta} \Phi^{\text{op}} T_1, \)

\[
\begin{array}{ccc}
X_1 & \xrightarrow{F} & X_2 \\
T_1 & \downarrow & T_2 \\
B_1^{\text{op}} & \xrightarrow{\Phi^{\text{op}}} & B_2^{\text{op}}
\end{array}
\]

- **composition** of two affine theories \( T_1 \xrightarrow{\eta_1} T_2, \ T_2 \xrightarrow{\eta_2} T_3 \) is \( T_3 F_2 F_1 \xrightarrow{\eta_2 \circ \eta_1} \Phi_2^{\text{op}} \Phi_1^{\text{op}} T_1 = T_3 F_2 F_1 \xrightarrow{\eta_2 F_1} \Phi_2^{\text{op}} T_2 F_1 \xrightarrow{\Phi_2^{\text{op}} \eta_1} \Phi_2^{\text{op}} \Phi_1^{\text{op}} T_1; \)

- **identity** on a theory \( T \) is the identity natural transformation \( T \xrightarrow{1_T} T. \)
Models of affine theories

**Definition 21**

\( \text{AfStm} \) is the category, whose objects are categories of the form \( \text{AfSys}(T) \) and whose morphisms are functors between them.

**Theorem 22**

\[
\begin{align*}
\text{AfTh} & \xrightarrow{\text{AfSys}} \text{AfStm}, \quad \text{AfSys}(T_1 \xrightarrow{\eta} T_2) = \text{AfSys}(T_1) \\
\text{AfSys}(T_2), \quad \text{AfSys}\eta((X, \kappa, B) \xrightarrow{(f, \phi)} (X', \kappa', B')) = (FX, \Phi^{\text{op}}\kappa \circ \eta_X, \Phi^{\text{op}}B) \xrightarrow{(Ff, \Phi^{\text{op}}\phi)} (FX', \Phi^{\text{op}}\kappa' \circ \eta_{X'}, \Phi^{\text{op}}B')
\end{align*}
\]

is a functor.

The respective functor for affine spaces requires more effort.
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\[
\text{AfTh} \xrightarrow{\text{AfSys}} \text{AfStm}, \quad \text{AfSys}(T_1 \xrightarrow{\eta} T_2) = \text{AfSys}(T_1) \xrightarrow{\text{AfSys}\eta} \text{AfSys}(T_2), \quad \text{AfSys}\eta((X, \kappa, B) \xrightarrow{(f, \varphi)} (X', \kappa', B')) = (FX, \Phi^{\text{op}}\kappa \circ \eta_X, \Phi^{\text{op}}B) \xrightarrow{(Ff, \Phi^{\text{op}}\varphi)} (FX', \Phi^{\text{op}}\kappa' \circ \eta_{X'}, \Phi^{\text{op}}B') \text{ is a functor.}
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$\text{AfStm}$ is the category, whose objects are categories of the form $\text{AfSys}(T)$ and whose morphisms are functors between them.

**Theorem 22**

$\text{AfTh} \xrightarrow{\text{AfSys}} \text{AfStm}$, $\text{AfSys}(T_1 \xrightarrow{\eta} T_2) = \text{AfSys}(T_1) \xrightarrow{\text{AfSys}\eta} \text{AfSys}(T_2)$, $\text{AfSys}\eta((X,\kappa,B) \xrightarrow{(f,\varphi)} (X',\kappa',B')) = (FX,\Phi^{op}\kappa \circ \eta X,\Phi^{op}B)$ is a functor.

The respective functor for affine spaces requires more effort.
Institutions

Definition 23

An institution \( \mathcal{I} \) consists of:

- a category \( \text{Sign} \) of signatures, \( \Sigma \) denoting an arbitrary object,
- a functor \( \text{Sign} \xrightarrow{\text{Mod}} \text{Cat}^{\text{op}} \) giving \( \Sigma \)-models and \( \Sigma \)-morphisms,
- a functor \( \text{Sign} \xrightarrow{\text{Sen}} \text{Cat} \) giving \( \Sigma \)-sentences and \( \Sigma \)-proofs,
- a satisfaction relation \( \models_{\Sigma} \subseteq \text{Ob}(\text{Mod}\Sigma) \times \text{Ob}(\text{Sen}\Sigma) \) for every \( \Sigma \in \text{Ob}(\text{Sign}) \)

such that

satisfaction: \( m' \models_{\Sigma} \text{Sen}\phi(s) \iff \text{Mod}\phi(m') \models_{\Sigma} s \) for every \( m' \in \text{Ob}(\text{Mod}\Sigma') \), \( s \in \text{Ob}(\text{Sen}\Sigma) \), \( \Sigma \xrightarrow{\phi} \Sigma' \) in \( \text{Sign} \),

soundness: \( m \models_{\Sigma} s \) and \( s \rightarrow s' \) in \( \text{Sen}\Sigma \) imply \( m \models_{\Sigma} s' \) for \( m \in \text{Ob}(\text{Mod}\Sigma) \).
**Definition 24**

An *institution morphism* \( \mathcal{I} \xrightarrow{(\Phi, \alpha, \beta)} \mathcal{I}' \) comprises

- a functor \( \text{Sign} \xrightarrow{\Phi} \text{Sign}' \),
- natural transformations \( \text{Sen}' \Phi \xrightarrow{\alpha} \text{Sen} \) and \( \text{Mod} \xrightarrow{\beta} \text{Mod}' \Phi \),

such that the following *satisfaction condition* holds

\[
m \models \Sigma (s') \iff \beta_{\Sigma} (m) \models'_{\Phi_{\Sigma}} s'
\]

for every \( \Sigma \)-model \( m \) from \( \mathcal{I} \) and every \( \Phi_{\Sigma} \)-sentence \( s' \) from \( \mathcal{I}' \).
Elementary institutions

Definition 25

- An institution is called *elementary* provided that the category \( \text{Cat} \) is replaced with the category \( \text{Set} \).
- \( \text{Inst} \) (resp. \( \text{EllInst} \)) is the category of (resp. elementary) institutions and their morphisms.
Topological institutions and their morphisms

Definition 26

- A *topological institution* is a functor $\text{Sign} \xrightarrow{T} \text{TopSys}^{op}$, where $\text{Sign}$ is a category of (abstract) signatures.

- A *topological institution morphism* $(\text{Sign}, T) \xrightarrow{(\Phi, \alpha)} (\text{Sign}', T')$ consists of a functor $\text{Sign} \xrightarrow{\Phi} \text{Sign}'$ and a natural transformation $T \xrightarrow{\alpha} T'\Phi$.

- $\text{TpInst}$ is the category of topological institutions and their morphisms.
Affine institutions and their morphisms

Definition 27

- An **affine institution** is a functor $S \xrightarrow{l} \text{AfSys}(T)$, where $S$ is a category of (abstract) signatures.

- An **affine institution morphism** $(S_1, l_1, T_1) \xrightarrow{(\Phi, \alpha, \eta)} (S_2, l_2, T_2)$ comprises a functor $S_1 \xrightarrow{\Phi} S_2$, an affine theory morphism $T_1 \xrightarrow{\eta} T_2$, and a natural transformation $\text{AfSys} \eta l_1 \xrightarrow{\alpha} l_2 \Phi$.

$$
\begin{array}{ccc}
S_1 & \xrightarrow{\Phi} & S_2 \\
\downarrow{l_1} & & \downarrow{l_2} \\
\text{AfSys}(T_1) & \xrightarrow{\text{AfSys} \eta} & \text{AfSys}(T_2).
\end{array}
$$

- **AflInst** is the category of affine institutions and their morphisms.
Affine institutions and their morphisms

Examples of affine institutions

**Definition 28**

Given an affine theory $T$, $\text{AflInst}(T)$ stands for the subcategory of $\text{AflInst}$ consisting of affine institutions $(S, I, T)$ (shortened to $(S, I)$) and their respective morphisms $(\Phi, \alpha, 1_T)$ (shortened to $(\Phi, \alpha)$).

**Example 29**

1. For $B = \text{Frm}$, $\text{AflInst}(\mathcal{P}_2)$ is a modification of $\text{TpInst}$.

2. $\text{Set} \xrightarrow{|\mathcal{P}_2|} \text{Set}^{op} := \text{Set} \xrightarrow{\mathcal{P}_3} \text{CBAlg}^{op} \xrightarrow{|I|^{op}} \text{Set}^{op}$ gives the category $\text{AflInst}(|\mathcal{P}_2|)$, which is a modification of $\text{EllInst}$. 
Examples of affine institutions

Definition 28

Given an affine theory $T$, $\text{AflInst}(T)$ stands for the subcategory of $\text{AflInst}$ consisting of affine institutions $(S, I, T)$ (shortened to $(S, I)$) and their respective morphisms $(\Phi, \alpha, 1_T)$ (shortened to $(\Phi, \alpha)$).

Example 29

1. For $B = \text{Frm}$, $\text{AflInst}(\mathcal{P}_2)$ is a modification of $\text{TplInst}$.
2. $\text{Set} \xrightarrow{|\mathcal{P}_2|} \text{Set}^{op} := \text{Set} \xrightarrow{\mathcal{P}_2} \text{CBAlg}^{op} \xrightarrow{|-|^{op}} \text{Set}^{op}$ gives the category $\text{AflInst}(|\mathcal{P}_2|)$, which is a modification of $\text{EllInst}$. 
Spatial affine institutions

**Definition 30**

Let $T$ be an affine theory.

- A *spatial affine $T$-institution* is a functor $S \xrightarrow{I} \text{AfSpc}(T)$, where $S$ is a category of (abstract) signatures.

- A *spatial affine $T$-institution morphism* $(S_1, I_1) \xrightarrow{(\Phi, \alpha)} (S_2, I_2)$ comprises a functor $S_1 \xrightarrow{\Phi} S_2$ and a natural transformation $I_1 \xrightarrow{\alpha} I_2 \Phi$.

- $\text{SAfInst}(T)$ is the category of spatial affine $T$-institutions and their morphisms.
Theorem 31

1. \( \text{SAfInst}(T) \xleftarrow{IE} \text{AfInst}(T), \quad IE((S_1, I_1) \xrightarrow{(\Phi, \alpha)} (S_2, I_2)) = (S_1, EI_1) \xrightarrow{(\Phi, E\alpha)} (S_2, EI_2) \) is a full embedding.

2. \( \text{AfInst}(T) \xrightarrow{ISpat} \text{SAfInst}(T), \quad ISpat((S_1, I_1) \xrightarrow{(\Phi, \alpha)} (S_2, I_2)) = (S_1, SpatI_1) \xrightarrow{(\Phi, Spat\alpha)} (S_2, SpatI_2) \) is a right-adjoint-left-inverse to \( IE \).

3. \( \text{SAfInst}(T) \) is isomorphic to a full coreflective subcategory of \( \text{AfInst}(T) \).

This answers the question on spatialization construction for topological institutions of A. Sernadas, C. Sernadas, and J. M. Valença.
Theorem 31

1. \( \text{SAfInst}(T) \xleftarrow{IE} \text{AfInst}(T), \quad IE \left( (S_1, I_1) \xrightarrow{(\Phi, \alpha)} (S_2, I_2) \right) = (S_1, EI_1) \xrightarrow{(\Phi, E\alpha)} (S_2, EI_2) \) is a full embedding.

2. \( \text{AfInst}(T) \xrightarrow{ISpat} \text{SAfInst}(T), \quad ISpat \left( (S_1, I_1) \xrightarrow{(\Phi, \alpha)} (S_2, I_2) \right) = (S_1, Spatl_1) \xrightarrow{(\Phi, Spat\alpha)} (S_2, Spatl_2) \) is a right-adjoint-left-inverse to \( IE \).

3. \( \text{SAfInst}(T) \) is isomorphic to a full coreflective subcategory of \( \text{AfInst}(T) \).

This answers the question on spatialization construction for topological institutions of A. Sernadas, C. Sernadas, and J. M. Valença.
**Definition 32**

Let $X \xrightarrow{T} B^{\text{op}}$ be an affine theory.

- A **localic affine $T$-institution** is a functor $S \xrightarrow{I} B^{\text{op}}$, where $S$ is a category of (abstract) signatures.

- A **localic affine $T$-institution morphism** $(S_1, I_1) \xrightarrow{(\Phi, \alpha)} (S_2, I_2)$ comprises a functor $S_1 \xrightarrow{\Phi} S_2$ and a natural transformation $I_1 \xrightarrow{\alpha} I_2 \Phi$.

- $\text{LAfInst}(T)$ is the category of localic affine $T$-institutions and their morphisms.
Theorem 33

Let $T$ be an affine theory such that there exists an adjoint situation $(\eta, \varepsilon) : T \vdash Pt : \mathcal{B}^{op} \rightarrow \mathbf{X}$.

1. $\text{LAfInst}(T) \xleftarrow{IE} \text{AflInst}(T),\ IE((S_1, I_1) \xrightarrow{(\Phi, \alpha)} (S_2, I_2)) = (S_1, EI_1) \xrightarrow{(\Phi, E\alpha)} (S_2, EI_2)$ is a full embedding.

2. $IE$ has a left-adjoint-left-inverse $\text{AflInst}(T) \xrightarrow{ILoc} \text{LAfInst}(T),\ ILoc((S_1, I_1) \xrightarrow{(\Phi, \alpha)} (S_2, I_2)) = (S_1, Locl_1) \xrightarrow{(\Phi, Loc\alpha)} (S_2, Locl_2)$.

3. $\text{LAfInst}(T)$ is isomorphic to a full reflective subcategory of $\text{AflInst}(T)$.
Conclusion

- Following the concept of topological institution, we introduced the notion of affine institution and showed its respective spatialization and localification procedures.
- Affine institutions seem to provide a good framework for elementary institutions and topological institutions, since they do not require the employed algebraic structures to be frames.
- While A. Sernadas, C. Sernadas, and J. M. Valença impose the frame structure on the set of theories (certain “closed” subsets of the set of sentences) of a given signature, which results in technical difficulties, we suggest the use of an arbitrary algebraic structure, which could be determined in each concrete case.
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References II


Thank you for your attention!