Using topological systems to create a framework for institutions

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Outline

Institutions

- There is a convenient approach to logical systems in computer science based in *institutions* of J. A. Goguen and R. M. Burstall.
- An institution comprises a category of (abstract) signatures, where every signature has its associated sentences, models, and a relation of satisfaction, which is invariant under change of signature, i.e., "truth is invariant under change of notation".
- Institutions include unsorted universal algebra, many-sorted algebra, order-sorted algebra, first-order logic, partial algebra.
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- Based in the ideas of geometric logic, *topological systems* of S. Vickers provide a common setting for both topological spaces and their underlying algebraic structures—locales.
- S. Vickers showed system spatialization and localification procedures, i.e., ways to move back and forth between the categories of topological spaces (resp., locales) and topological systems.
- Recently, topological systems have gained in interest in connection with lattice-valued topology, e.g., one has
  - introduced and studied lattice-valued topological systems;
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Institutions versus systems

- To find relationships between institutions and topological systems, J. T. Denniston, A. Melton, and S. E. Rodabaugh introduced *lattice-valued institutions*, and showed that lattice-valued topological systems provide their particular instance.
- A. Sernadas, C. Sernadas, and J. M. Valença introduced crisp *topological institutions* based in topological systems, the slogan being that "the central concept is the theory, not the formula".
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Ω-algebras and Ω-homomorphisms

Definition 1

Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a family of cardinal numbers, which is indexed by a (possibly proper or empty) class Λ.

- An *Ω-algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$, comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (*n_λ-ary primitive operations* on A).
- An *Ω-homomorphism* $(A_1, (\omega_\lambda^{A_1})_{\lambda \in \Lambda}) \xrightarrow{\varphi} (A_2, (\omega_\lambda^{A_2})_{\lambda \in \Lambda})$ is a map $A_1 \xrightarrow{\varphi} A_2$ such that $\varphi \circ \omega_\lambda^{A_1} = \omega_\lambda^{A_2} \circ \varphi^{n_\lambda}$ for every $\lambda \in \Lambda$.
- $\mathbf{Alg}(\Omega)$ is the construct of Ω-algebras and Ω-homomorphisms.

Varieties and algebras

Definition 2

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety of Ω-algebras* is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients, and whose objects (resp. morphisms) are called *algebras* (resp. *homomorphisms*).

Examples of varieties

Example 3

1. **CSLat**($\bigvee$) is the variety of $\bigvee$-*semilattices*, and **CSLat**($\bigwedge$) is the variety of $\bigwedge$-*semilattices*.
2. **Frm** is the variety of *frames*.
3. **CBAlg** is the variety of *complete Boolean algebras*.
4. **CSL** is the variety of *closure semilattices*, i.e., $\bigwedge$-semilattices, with the singled out bottom element.

Affine spaces

The following extends the notion of *affine set* of Y. Diers.

Definition 4

Given a functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, where $\mathbf{B}$ is a variety of algebras, $\mathbf{AfSpc}(T)$ is the concrete category over $\mathbf{X}$, whose

objects (*T-affine spaces* or *T-spaces*) are pairs (X, τ), where X is an $\mathbf{X}$-object and τ is a subalgebra of TX;

morphisms (*T-affine morphisms* or *T-morphisms*) $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ are $\mathbf{X}$-morphisms $X_1 \xrightarrow{f} X_2$ with the property that $(Tf)^{op}(\alpha) \in \tau_1$ for every $\alpha \in \tau_2$.

The concrete category $(\mathbf{AfSpc}(T), |-|)$ is topological over $\mathbf{X}$.

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Example 5

Given a variety $\mathbf{B}$, every subcategory $\mathbf{S}$ of $\mathbf{B}^{op}$ induces a functor $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{P}_{\mathbf{S}}} \mathbf{B}^{op}$, $\mathcal{P}_{\mathbf{S}}((X_1, B_1) \xrightarrow{(f,\varphi)} (X_2, B_2)) = B_1^{X_1} \xrightarrow{\mathcal{P}_{\mathbf{S}}(f,\varphi)} B_2^{X_2}$, where $(\mathcal{P}_{\mathbf{S}}(f,\varphi))^{op}(\alpha) = \varphi^{op} \circ \alpha \circ f$.

Example 6

1. If $\mathbf{B} = \mathbf{Frm}$, then $\mathbf{AfSpc}(\mathcal{P}_2) = \mathbf{Top}$ (topological spaces).
2. If $\mathbf{B} = \mathbf{CSL}$, then $\mathbf{AfSpc}(\mathcal{P}_2) = \mathbf{Cls}$ (closure spaces).
3. $\mathbf{AfSpc}(\mathcal{P}_B)$ is the category $\mathbf{AfSet}(B)$ of affine sets of Y. Diers.
4. If $\mathbf{B} = \mathbf{Frm}$, then $\mathbf{AfSpc}(\mathcal{P}_{\mathbf{S}}) = \mathbf{S}\text{-}\mathbf{Top}$ (variable-basis lattice-valued topological spaces of S. E. Rodabaugh).

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Affine systems

Definition 7

Given a functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, $\mathbf{AfSys}(T)$ is the comma category $(T \downarrow 1_{\mathbf{B}^{op}})$, concrete over the product category $\mathbf{X} \times \mathbf{B}^{op}$, whose

objects (T*-affine systems* or T*-systems*) are triples (X, κ, B), made by $\mathbf{B}^{op}$-morphisms $TX \xrightarrow{\kappa} B$;

morphisms (T*-affine morphisms* or T*-morphisms*) $(X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, B_2)$ are $\mathbf{X} \times \mathbf{B}^{op}$-morphisms $(X_1, B_1) \xrightarrow{(f,\varphi)} (X_2, B_2)$, making the next diagram commute

$$\begin{array}{ccc} TX_1 & \xrightarrow{Tf} & TX_2 \\ {\scriptstyle \kappa_1}\downarrow & & \downarrow{\scriptstyle \kappa_2} \\ B_1 & \xrightarrow[\varphi]{} & B_2. \end{array}$$

Examples

Example 8

1. If $\mathbf{B} = \mathbf{Frm}$, then $\mathbf{AfSys}(\mathcal{P}_2) = \mathbf{TopSys}$ (topological systems of S. Vickers).
2. If $\mathbf{B} = \mathbf{Set}$, then $\mathbf{AfSys}(\mathcal{P}_B) = \mathbf{Chu}_B$ (Chu spaces over a set B of P.-H. Chu).

Definition 9

A T-system (X, κ, B) is called *separated* provided that $TX \xrightarrow{\kappa} B$ is an epimorphism in $\mathbf{B}^{op}$. $\mathbf{AfSys}_s(T)$ is the full subcategory of $\mathbf{AfSys}(T)$ of separated T-systems.

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For $\mathbf{B} = \mathbf{CSL}$, $\mathbf{AfSys}_s(\mathcal{P}_2) = \mathbf{SP}$ (state property systems of D. Aerts).

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Affine spatialization procedure

Theorem 11

1. $\mathbf{AfSpc}(T) \overset{E}{\hookrightarrow} \mathbf{AfSys}(T)$, $E((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1, e^{op}_{\tau_1}, \tau_1) \xrightarrow{(f,\varphi)} (X_2, e^{op}_{\tau_2}, \tau_2)$ *is a full embedding, with* e_{τ_i} *the inclusion* $\tau_i \hookrightarrow TX_i$ *and* φ^{op} *the restriction* $\tau_2 \xrightarrow{(Tf)^{op}|^{\tau_1}_{\tau_2}} \tau_1$.
2. E *has a right-adjoint-left-inverse* $\mathbf{AfSys}(T) \xrightarrow{Spat} \mathbf{AfSpc}(T)$, $Spat((X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, B_2)) = (X_1, \kappa^{op}_1(B_1)) \xrightarrow{f} (X_2, \kappa^{op}_2(B_2))$.
3. $\mathbf{AfSpc}(T)$ *is isomorphic to a full (regular mono)-coreflective subcategory of* $\mathbf{AfSys}(T)$.

Consequences

Theorem 12

E and Spat restrict to $\mathbf{AfSpc}(T) \overset{\overline{E}}{\hookrightarrow} \mathbf{AfSys}_s(T)$ *and* $\mathbf{AfSys}_s(T) \xrightarrow{\overline{Spat}} \mathbf{AfSpc}(T)$*, providing an equivalence between the categories* $\mathbf{AfSpc}(T)$ *and* $\mathbf{AfSys}_s(T)$ *such that* $\overline{Spat}\,\overline{E} = 1_{\mathbf{AfSpc}(T)}$.

Corollary 13

$\mathbf{AfSpc}(T)$ *is the amnestic modification of* $\mathbf{AfSys}_s(T)$.

Example 14

1. **Top** is isomorphic to a full (regular mono)-coreflective subcategory of **TopSys** (system spatialization procedure of S. Vickers).
2. The categories **Cls** and **SP** are equivalent.

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Affine localification procedure

Proposition 15

$\mathbf{AfSys}(T) \xrightarrow{Loc} \mathbf{B}^{op}$, $Loc((X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, B_2)) = B_1 \xrightarrow{\varphi} B_2$ *is a functor.*

Theorem 16

Given a functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, *the following are equivalent.*

1. *There exists an adjoint situation* $(\eta, \varepsilon) : T \dashv Pt : \mathbf{B}^{op} \to \mathbf{X}$.
2. *There exists a full embedding* $\mathbf{B}^{op} \overset{E}{\hookrightarrow} \mathbf{AfSys}(T)$ *such that* Loc *is a left-adjoint-left-inverse to* E. $\mathbf{B}^{op}$ *is then isomorphic to a full reflective subcategory of* $\mathbf{AfSys}(T)$.

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Examples

Remark 17

Every functor $\mathbf{Set} \xrightarrow{\mathcal{P}_B} \mathbf{B}^{op}$ *has a right adjoint* $\mathbf{B}^{op} \xrightarrow{Pt_B} \mathbf{Set}$, $Pt_B(B_1 \xrightarrow{\varphi} B_2) = \mathbf{B}(B_1, B) \xrightarrow{Pt_B\varphi} \mathbf{B}(B_2, B)$, $(Pt_B\varphi)(p) = p \circ \varphi^{op}$.

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- **Loc** is isomorphic to a full reflective subcategory of **TopSys**, which gives the system localification procedure of S. Vickers.
- $\mathbf{B}^{op}$ is isomorphic to a full reflective subcategory of $\mathbf{AfSys}(\mathcal{P}_B)$.

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Affine theories

One could like to study the properties of the categories **AfSys**(T) and **AfSpc**(T) through the properties of the functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$.

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An *affine theory* is a functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ with $\mathbf{B}$ a variety of algebras.

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Category of affine theories

Definition 20

AfTh is the category given by the following data:

objects are affine theories $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$;

morphisms $T_1 \xrightarrow{(F,\Phi,\eta)} T_2$ (shortened to η) comprise two functors $\mathbf{X}_1 \xrightarrow{F} \mathbf{X}_2$, $\mathbf{B}_1 \xrightarrow{\Phi} \mathbf{B}_2$ and a natural transformation $T_2 F \xrightarrow{\eta} \Phi^{op} T_1$,

$$\begin{array}{ccc} \mathbf{X}_1 & \xrightarrow{\quad F \quad} & \mathbf{X}_2 \\ {\scriptstyle T_1}\downarrow & \overset{\eta}{\swarrow\!\!\!\!\swarrow} & \downarrow{\scriptstyle T_2} \\ \mathbf{B}_1^{op} & \xrightarrow{\quad \Phi^{op} \quad} & \mathbf{B}_2^{op} \end{array}$$

composition of two affine theories $T_1 \xrightarrow{\eta_1} T_2$, $T_2 \xrightarrow{\eta_2} T_3$ is $T_3 F_2 F_1 \xrightarrow{\eta_2 \odot \eta_1} \Phi_2^{op} \Phi_1^{op} T_1 = T_3 F_2 F_1 \xrightarrow{\eta_2 F_1} \Phi_2^{op} T_2 F_1 \xrightarrow{\Phi_2^{op} \eta_1} \Phi_2^{op} \Phi_1^{op} T_1$;

identity on a theory T is the identity natural transformation $T \xrightarrow{1_T} T$.

Models of affine theories

Definition 21

AfStm is the category, whose objects are categories of the form **AfSys**(T) and whose morphisms are functors between them.

Theorem 22

$\mathbf{AfTh} \xrightarrow{AfSys} \mathbf{AfStm}$, $AfSys(T_1 \xrightarrow{\eta} T_2) = \mathbf{AfSys}(T_1) \xrightarrow{AfSys\eta} \mathbf{AfSys}(T_2)$, $AfSys\eta((X, \kappa, B) \xrightarrow{(f,\varphi)} (X', \kappa', B')) = (FX, \Phi^{op}\kappa \circ \eta_X, \Phi^{op}B) \xrightarrow{(Ff,\Phi^{op}\varphi)} (FX', \Phi^{op}\kappa' \circ \eta_{X'}, \Phi^{op}B')$ is a functor.

The respective functor for affine spaces requires more effort.

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Institutions

Definition 23

An *institution* $\mathbb{I}$ consists of:

- a category **Sign** of *signatures*, Σ denoting an arbitrary object,
- a functor $\mathbf{Sign} \xrightarrow{Mod} \mathbf{Cat}^{op}$ giving *Σ-models* and *Σ-morphisms*,
- a functor $\mathbf{Sign} \xrightarrow{Sen} \mathbf{Cat}$ giving *Σ-sentences* and *Σ-proofs*,
- a *satisfaction* relation $\models_\Sigma \subseteq Ob(Mod\Sigma) \times Ob(Sen\Sigma)$ for every $\Sigma \in Ob(\mathbf{Sign})$

such that

satisfaction: $m' \models_{\Sigma'} Sen\phi(s)$ iff $Mod\phi(m') \models_\Sigma s$ for every $m' \in Ob(Mod\Sigma')$, $s \in Ob(Sen\Sigma)$, $\Sigma \xrightarrow{\phi} \Sigma'$ in **Sign**,

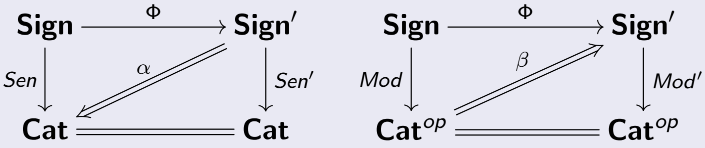
soundness: $m \models_\Sigma s$ and $s \to s'$ in $Sen\Sigma$ imply $m \models_\Sigma s'$ for $m \in Ob(Mod\Sigma)$.

Institution morphisms

Definition 24

An *institution morphism* $\mathbb{I} \xrightarrow{(\Phi,\alpha,\beta)} \mathbb{I}'$ comprises

- a functor $\mathbf{Sign} \xrightarrow{\Phi} \mathbf{Sign}'$,
- natural transformations $Sen'\Phi \xrightarrow{\alpha} Sen$ and $Mod \xrightarrow{\beta} Mod'\Phi$,



such that the following *satisfaction condition* holds

$$m \models_{\Sigma} \alpha_{\Sigma}(s') \text{ iff } \beta_{\Sigma}(m) \models'_{\Phi\Sigma} s'$$

for every Σ-model m from $\mathbb{I}$ and every $\Phi\Sigma$-sentence s' from $\mathbb{I}'$.

Elementary institutions

Definition 25

- An institution is called *elementary* provided that the category **Cat** is replaced with the category **Set**.
- **Inst** (resp. **ElInst**) is the category of (resp. elementary) institutions and their morphisms.

Topological institutions and their morphisms

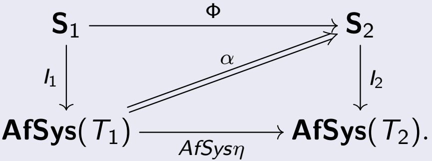
Definition 26

- A *topological institution* is a functor $\mathbf{Sign} \xrightarrow{\mathcal{T}} \mathbf{TopSys}^{op}$, where $\mathbf{Sign}$ is a category of (abstract) signatures.
- A *topological institution morphism* $(\mathbf{Sign}, \mathcal{T}) \xrightarrow{(\Phi,\alpha)} (\mathbf{Sign}', \mathcal{T}')$ consists of a functor $\mathbf{Sign} \xrightarrow{\Phi} \mathbf{Sign}'$ and a natural transformation $\mathcal{T} \xrightarrow{\alpha} \mathcal{T}'\Phi$.
- **TpInst** is the category of topological institutions and their morphisms.

Affine institutions and their morphisms

Definition 27

- An *affine institution* is a functor $\mathbf{S} \xrightarrow{I} \mathbf{AfSys}(T)$, where $\mathbf{S}$ is a category of (abstract) signatures.
- An *affine institution morphism* $(\mathbf{S}_1, I_1, T_1) \xrightarrow{(\Phi,\alpha,\eta)} (\mathbf{S}_2, I_2, T_2)$ comprises a functor $\mathbf{S}_1 \xrightarrow{\Phi} \mathbf{S}_2$, an affine theory morphism $T_1 \xrightarrow{\eta} T_2$, and a natural transformation $AfSys\eta I_1 \xrightarrow{\alpha} I_2\Phi$,



- **AfInst** is the category of affine institutions and their morphisms.

Examples of affine institutions

Definition 28

Given an affine theory T, **AfInst**(T) stands for the subcategory of **AfInst** consisting of affine institutions $(\mathbf{S}, I, T)$ (shortened to $(\mathbf{S}, I)$) and their respective morphisms $(\Phi, \alpha, 1_T)$ (shortened to (Φ, α)).

Example 29

1. For $\mathbf{B} = \mathbf{Frm}$, $\mathbf{AfInst}(\mathcal{P}_2)$ is a modification of **TpInst**.
2. $\mathbf{Set} \xrightarrow{|\mathcal{P}_2|} \mathbf{Set}^{op} := \mathbf{Set} \xrightarrow{\mathcal{P}_2} \mathbf{CBAlg}^{op} \xrightarrow{|-|^{op}} \mathbf{Set}^{op}$ gives the category $\mathbf{AfInst}(|\mathcal{P}_2|)$, which is a modification of **ElInst**.

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Spatial affine institutions

Definition 30

Let T be an affine theory.

- A *spatial affine T-institution* is a functor $\mathbf{S} \xrightarrow{I} \mathbf{AfSpc}(T)$, where $\mathbf{S}$ is a category of (abstract) signatures.
- A *spatial affine T-institution morphism* $(\mathbf{S}_1, I_1) \xrightarrow{(\Phi,\alpha)} (\mathbf{S}_2, I_2)$ comprises a functor $\mathbf{S}_1 \xrightarrow{\Phi} \mathbf{S}_2$ and a natural transformation $I_1 \xrightarrow{\alpha} I_2\Phi$.
- $\mathbf{SAfInst}(T)$ is the category of spatial affine T-institutions and their morphisms.

Affine institution spatialization procedure

Theorem 31

1. $\mathbf{SAfInst}(T) \overset{IE}{\hookrightarrow} \mathbf{AfInst}(T)$, $IE((\mathbf{S}_1, I_1) \xrightarrow{(\Phi,\alpha)} (\mathbf{S}_2, I_2)) = (\mathbf{S}_1, EI_1) \xrightarrow{(\Phi,E\alpha)} (\mathbf{S}_2, EI_2)$ *is a full embedding.*
2. $\mathbf{AfInst}(T) \xrightarrow{ISpat} \mathbf{SAfInst}(T)$, $ISpat((\mathbf{S}_1, I_1) \xrightarrow{(\Phi,\alpha)} (\mathbf{S}_2, I_2)) = (\mathbf{S}_1, SpatI_1) \xrightarrow{(\Phi,Spat\alpha)} (\mathbf{S}_2, SpatI_2)$ *is a right-adjoint-left-inverse to* IE.
3. $\mathbf{SAfInst}(T)$ *is isomorphic to a full coreflective subcategory of* $\mathbf{AfInst}(T)$.

This answers the question on spatialization construction for topological institutions of A. Sernadas, C. Sernadas, and J. M. Valença.

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This answers the question on spatialization construction for topological institutions of A. Sernadas, C. Sernadas, and J. M. Valença.

Localic affine institutions

Definition 32

Let $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ be an affine theory.

- A *localic affine T-institution* is a functor $\mathbf{S} \xrightarrow{I} \mathbf{B}^{op}$, where $\mathbf{S}$ is a category of (abstract) signatures.
- A *localic affine T-institution morphism* $(\mathbf{S}_1, I_1) \xrightarrow{(\Phi,\alpha)} (\mathbf{S}_2, I_2)$ comprises a functor $\mathbf{S}_1 \xrightarrow{\Phi} \mathbf{S}_2$ and a natural transformation $I_1 \xrightarrow{\alpha} I_2\Phi$.
- $\mathbf{LAfInst}(T)$ is the category of localic affine T-institutions and their morphisms.

Affine institution localification procedure

Theorem 33

Let T be an affine theory such that there exists an adjoint situation $(\eta, \varepsilon) : T \dashv Pt : \mathbf{B}^{op} \to \mathbf{X}$.

1. $\mathbf{LAfInst}(T) \overset{IE}{\hookrightarrow} \mathbf{AfInst}(T)$, $IE((\mathbf{S}_1, I_1) \xrightarrow{(\Phi,\alpha)} (\mathbf{S}_2, I_2)) = (\mathbf{S}_1, EI_1) \xrightarrow{(\Phi,E\alpha)} (\mathbf{S}_2, EI_2)$ *is a full embedding.*
2. *IE has a left-adjoint-left-inverse* $\mathbf{AfInst}(T) \xrightarrow{ILoc} \mathbf{LAfInst}(T)$, $ILoc((\mathbf{S}_1, I_1) \xrightarrow{(\Phi,\alpha)} (\mathbf{S}_2, I_2)) = (\mathbf{S}_1, LocI_1) \xrightarrow{(\Phi,Loc\alpha)} (\mathbf{S}_2, LocI_2)$.
3. $\mathbf{LAfInst}(T)$ *is isomorphic to a full reflective subcategory of* $\mathbf{AfInst}(T)$.

Conclusion

- Following the concept of topological institution, we introduced the notion of affine institution and showed its respective spatialization and localification procedures.
- Affine institutions seem to provide a good framework for elementary institutions and topological institutions, since they do not require the employed algebraic structures to be frames.
- While A. Sernadas, C. Sernadas, and J. M. Valença. impose the frame structure on the set of theories (certain "closed" subsets of the set of sentences) of a given signature, which results in technical difficulties, we suggest the use of an arbitrary algebraic structure, which could be determined in each concrete case.

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Thank you for your attention!