On weak constant domain principle in the Kripke sheaf semantics

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We consider superintuitionistic predicate logics understood in the usual way, as sets of predicate formulas (without equality and function symbols) containing all axioms of Heyting predicate logic **Q-H** and closed under modus ponens, generalization, and substitution of arbitrary formulas for atomic ones.

1 We consider the semantics of predicate Kripke frames with equality (called *e*-frames, for short), which is equivalent to the semantics of Kripke sheaves (see e.g. [1] or [2]). Namely, an *e-frame* is a triple $M = (W, U, I)$ formed by a poset W with the least element 0_W, a domain map U defined on W such that $\varnothing \neq U(u) \subseteq U(v)$ for $u \leq v$, and a family I of equivalence relations I_u on $U(u)$ for $u \in W$ such that $I_u \subseteq I_v$ for $u \leq v$. A usual *(predicate) Kripke frame* is an *e*-frame with equalities I_u (i.e., $aI_ub \Leftrightarrow a = b$ for $u \in W$, $a, b \in U(u)$).

A *valuation* $u \models A$ (for $u \in W$ and formulas A with parameters replaced by elements of $U(u)$) satisfies the monotonicity: $u \leq v,\ u \models A \ \Rightarrow\ v \models A$, the usual inductive clauses for connectives and quantifiers, e.g.

$u \models (B \to C) \ \Leftrightarrow\ \forall v \geq u\, [(v \models B) \Rightarrow (v \models C)]$,

$u \models \forall x\, B(x) \ \Leftrightarrow\ \forall v \geq u\, \forall c \in U(v)\, [v \models B(c)]$, etc.,

and preserves I_u (on every $U(u), u \in W$), i.e.,

$$\bigwedge_i (a_i\, I_u\, b_i) \ \Rightarrow\ (\, u \models A(a_1, \ldots, a_n) \ \Leftrightarrow\ u \models A(b_1, \ldots, b_n)\,).$$

A formula $A(\mathbf{x})$ (where $\mathbf{x} = (x_1, \ldots, x_n)$) is *valid* in M if it is true under any valuation in M, i.e., if $u \models A(\mathbf{a})$ for any $u \in W$ and $\mathbf{a} \in (D_u)^n$. The *predicate logic* $\mathbf{L}(M)$ *of an (e-)frame* M is the set of all formulas valid in M.

2 We consider the constant domain principle

$$D = \forall x(P(x) \vee Q) \to \forall x P(x) \vee Q$$

(where P and Q are unary and 0-ary symbols, respectively), and its weak ('negative') version

$$D^- = \forall x\,(\neg P(x) \vee Q) \to \forall x \neg P(x) \vee Q\,.$$

The formula D states (in an e-frame) that
$\forall a \in U(u)\, \exists b \in U(0_W)\,[\,a I_u b\,]$, and similarly, D^- states that
$\forall a \in U(u)\, \exists b \in U(0_W)\,[\,\exists v \geq u\,(\,a I_v b\,)\,]$.
Let D^--*frames* be e-frame validating D^-.
Clearly, $D \vdash D^-$ (we write $A \vdash B$ for $[\mathbf{Q}\text{-}\mathbf{H} + A] \vdash B$). Also:
D is valid in M iff D^- is valid in M iff
$U(u) = U(0_W)$ for every $u \in W$ for a usual Kripke frame M.
Hence the Kripke-completion of $[\mathbf{Q}\text{-}\mathbf{H} + D^-]$ is $[\mathbf{Q}\text{-}\mathbf{H} + D]$.
Now we describe the Kripke sheaf completion of $[\mathbf{Q}\text{-}\mathbf{H} + D^-]$.

3 We consider the following formulas (for $n>0$, $m\geq 0$):

$$\begin{aligned} D^-_{n,m} = \quad & \forall z(Q_0 \vee P_0(z))\&\forall x R(x,x) \ \rightarrow \\ & \rightarrow \ Q_0 \vee \forall \mathbf{x}_0 \, [\forall z(P_0(z) \rightarrow Q_1(\mathbf{x}_0) \vee P_1(\mathbf{x}_0, z)) \rightarrow \\ & \rightarrow Q_1(\mathbf{x}_0) \vee \forall \mathbf{x}_1 \, [\forall z(P_1(\mathbf{x}_0, z) \rightarrow Q_2(\mathbf{x}_0, \mathbf{x}_1) \vee P_2(\mathbf{x}_0, \mathbf{x}_1, z)) \rightarrow \\ & \rightarrow \ldots \\ & \rightarrow Q_{n-2}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-3}) \vee \forall \mathbf{x}_{n-2} \, [\forall z(P_{n-2}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-3}, z) \rightarrow \\ & \qquad \rightarrow \ Q_{n-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-2}) \vee P_{n-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-2}, z)) \rightarrow \\ & \rightarrow Q_{n-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-2}) \vee \forall \mathbf{x}_{n-1}, y \, [\forall z(P_{n-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-2}, z) \rightarrow \\ & \quad \rightarrow \ Q_n(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}, y) \vee \neg R(y, z)) \rightarrow Q_n(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}, y)]] \end{aligned}$$

Here P_i are $(1+m\cdot i)$-ary predicate symbols (for $0\leq i<n$), Q_i are $(m\cdot i)$-ary symbols (for $0\leq i<n$), Q_n is a $(1+m\cdot n)$-ary symbol, R is a binary symbol; also $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,m})$ (for $0\leq i<n$) are disjoint lists of different variables, and x, y, z are different variables non-occurring in $\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}$.

It can be easily shown that $D^-_{n,m} \vdash D^-_{n',m'}$ for $n \geq n', m \geq m'$ and $D^-_{1,0} \vdash D^-$. Moreover,

$$(\mathbf{Q}\text{-}\mathbf{H}+D^-) \subset (\mathbf{Q}\text{-}\mathbf{H}+\{D^-_{n,m} : n>0, m\geq 0\}) = (\mathbf{Q}\text{-}\mathbf{H}+\{D^-_{n,n} : n>0\}).$$

Also one can show that the formulas $D^-_{n,m}$ are valid in all D^--frames. Thus:
$D^-_{n,m}$ is valid in an e-frame M iff D^- is valid in an e-frame M, i.e., iff M is a D^--frame (for any n, m).

Theorem. The logic $(\mathbf{Q}\text{-}\mathbf{H} + \{D^-_{n,m} : n>0, m\geq 0\})$ is complete w.r.t. D^--frames.

Hence this logic is the Kripke sheaf completion of $(\mathbf{Q}\text{-}\mathbf{H} + D^-)$. We believe that this completion is not finitely axiomatizable.

Similar completeness results hold for extensions with:
1. Kuroda's formula $K = \neg\neg\forall x\,(P(x) \vee \neg P(x))$;
2. predicate axioms of finite heights P^+_m
(here $P^+_0 = \bot$ and $P^+_{n+1} = \forall x\,[\,R_n(x) \vee (R_n(x) \to P^+_n)\,]$ for $n \geq 0$;
R_n being different unary predicate symbols).

Bibliography

SHEHTMAN, V. and D. SKVORTSOV, *Semantics of non-classical first order predicate logics*. In: Mathematical Logic (ed. P.Petkov), Plenum Press, N.Y., 105–116, 1990 (Proc. of Summer school and conference in mathematical logic 'Heyting'88').

GABBAY, D., V. SHEHTMAN, and D. SKVORTSOV, *Quantification in nonclassical logic*, Vol. 1, Sections 2.6, 3.6, Studies in Logic and the Foundations of Mathematics 153: Elsevier, 2009.