Bisimulation games and locally tabular modal logics

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Modal propositional language

N-modal formulas are built from a countable set of

proposition letters PL={p_1,p_2,...} using boolean connectives

and unary modal connectives $\Box_1$,..., $\Box_N$;as usual $\Diamond_i = \neg \Box_i \neg$

If N=1 we denote the modalities just by $\Box$ and $\Diamond$.

The modal depth md(A) is defined by induction:

$md(p_i)=0$, $md(\neg A)=md(A)$,

$md(A \vee B)= md(A \wedge B) = max(md(A),md(B))$,

$md(\Box_i A)=md(A)+1$

Kripke frames and models-1

An N-modal Kripke frame is a nonempty set with N binary relations $F = (W, R_1, \ldots, R_N)$.

A valuation in F is a function $\theta : PL \to 2^W$ (so $\theta(p_i) \subseteq W$).

(F, θ) is a *Kripke model* over F.

In *k-weak Kripke models* only the letters $p_1, \ldots, p_k$ are evaluated.

Kripke frames and models-2

The inductive truth definition ($M,x \vDash A$) is standard.

- $M,x \vDash p_i$ iff $x \in \theta(p_i)$
- $M,x \vDash \Box_i A$ iff $\forall y(xR_i y \Rightarrow M,y \vDash A)$
- $M,x \vDash \Diamond_i A$ iff $\exists y(xR_i y \;\&\; M,y \vDash A)$

A formula A is valid in a frame F (in symbols, $F \vDash A$) if A is true at all points in every Kripke model over F.

Bisimulation games-1

Def For a k-weak Kripke model $M=(W,R_1,...,R_N,\theta)$ consider the *0-equivalence* relation between points

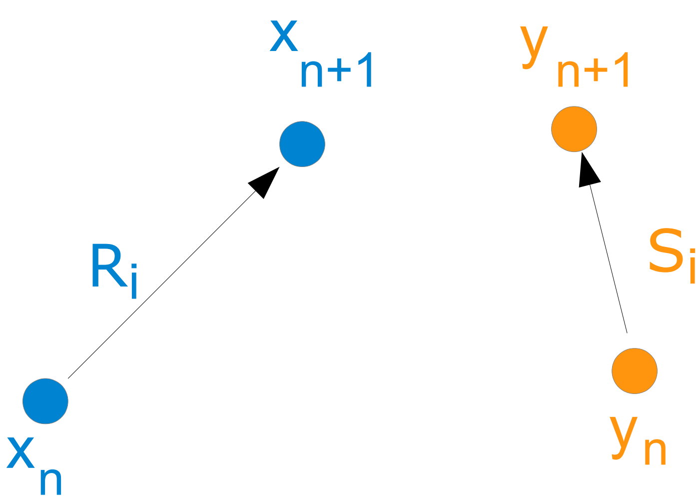
$$x \equiv_0 y := \forall j \leq k\ (M,x \models p_j \Leftrightarrow M,y \models p_j)$$

Given M and two points $x_0 \equiv_0 y_0$ we can play the *r-round bisimulation game* $BG_r(M,x_0,y_0)$.

Players: Spoiler (Abelard) vs Duplicator (Eloïse).

Bisimulation games-2

The initial position in $BG_r(M,x_0,M',y_0)$ is (x_0,y_0).



<u>Round (n+1)</u>

- Spoiler chooses i, x_{n+1} [or y_{n+1}] such that $x_n R_i x_{n+1}$ [$y_n R_i y_{n+1}$]
- Duplicator chooses y_{n+1} [x_{n+1}] such that $y_n R_i y_{n+1}$ [$x_n R_i x_{n+1}$] and $x_{n+1} \equiv_0 y_{n+1}$

- A player loses if he/she cannot move.
- Duplicator wins after r rounds.

Bisimulation games-3

<u>Def</u> Formula and game *n-equivalence* relations (on M)

- $x \equiv_n y$:= for any $A(p_1,\ldots,p_k)$ of modal depth $\leq n$

  $$M,x \vDash A \Leftrightarrow M',y \vDash A$$

- $x \sim_n y$:= Duplicator has a winning strategy in $BG_n(M,x,y)$

<u>Main Theorem on finite bisimulation games</u>

$$\equiv_n = \sim_n$$

Logics-1

We consider normal modal logics. An *N-modal logic* is a set of N-modal formulas

- containing all boolean tautologies,

$\Box_i(A \to B) \to (\Box_i A \to \Box_i B)$

- closed under Modus Ponens, Substitution, $\Box$-introduction

$(A/\Box_i A)$.

The *minimal logic* $\mathbf{K}_N$; $\mathbf{K} = \mathbf{K}_1$.

Logics-2

Kripke complete logics

$\mathbf{L}(F) := \{ A \mid F \vDash A \}$ (the *logic of a frame* F).

$\mathbf{L}(\mathcal{C}) := \bigcap \{\mathbf{L}(F) \mid F \in \mathcal{C}\}$ (the *logic of a class of frames* $\mathcal{C}$).

- If F is finite, $\mathbf{L}(F)$ is called *tabular* (or *finite*)
- If $\mathcal{C}$ consists of finite frames, $\mathbf{L}(\mathcal{C})$ has the finite model property (FMP). Or:

L has the FMP iff L is an intersection of tabular logics.

<u>Proposition</u> ('Harrop's theorem') If L is finitely axiomatizable and has the FMP, then L is decidable.

Modal algebras

An N-modal algebra is a Boolean algebra with an extra unary operations $\Box_1,\ldots,\Box_N$ satisfying the equality

$$\Box_i(x \cap y) = \Box_i x \cap \Box_i y.$$

Every modal formula A corresponds to an equality

$A^*=1$, where A^* is obtained by translating A into a term with Boolean operations and $\Box_i$.

A is called valid in an algebra $\mathfrak{A}$ if $A^*=1$ holds in $\mathfrak{A}$.

For a modal logic L, an L-algebra is a modal algebra validating L.

Formula depth-1

The *modal depth of a formula A in a modal logic L*

$md_L(A) := \min\{md(B) \mid L \vdash A \leftrightarrow B\}$

The *modal depth of a logic L*

$md(L) := \min\{md_L(A) \mid A \text{ is in the language of } L\}$

Formula depth-2

Canonical model theorem For any modal logic L (weak or not) one can construct the *canonical model* M_L such that for any A in the language of L

$$M_L \vDash A \text{ iff } L \vdash A$$

In every model we have a decreasing sequence $\equiv_0 \supseteq \equiv_1 \ldots$

$$\equiv_\infty := \bigcap_n \equiv_n$$

Formula depth-3

Lemma 1 Every set $W/\equiv_n$ (= $W/\sim_n$) is finite.

Lemma 2 $x \equiv_\infty y$ iff for any $A(p_1,\ldots,p_k)$ $(M,x \models A \Leftrightarrow M,y \models A)$

Lemma 3 In canonical models: $x \equiv_\infty y$ iff $x=y$.

Stabilization theorem If $\equiv_n \, = \, \equiv_{n+1}$ in every $M_{L\lceil k}$ (bisimulation games *stabilize at n*), then $md(L) \leq n$.

Local tabularity-1

$L\lceil k$ denotes the restriction of a logic L to formulas in variables $p_1,\ldots,p_k$. The sets $L\lceil k$ are called *weak modal logics*

Def A modal logic L is *locally tabular (or locally finite)* if for any k there are finitely many formulas in $p_1,\ldots,p_k$ up to equivalence in L.

Equivalently: A modal logic L is locally tabular if all its weak fragments $L\lceil k$ are tabular.

Local tabularity-2

Equivalent definitions of local tabularity for a modal logic L:

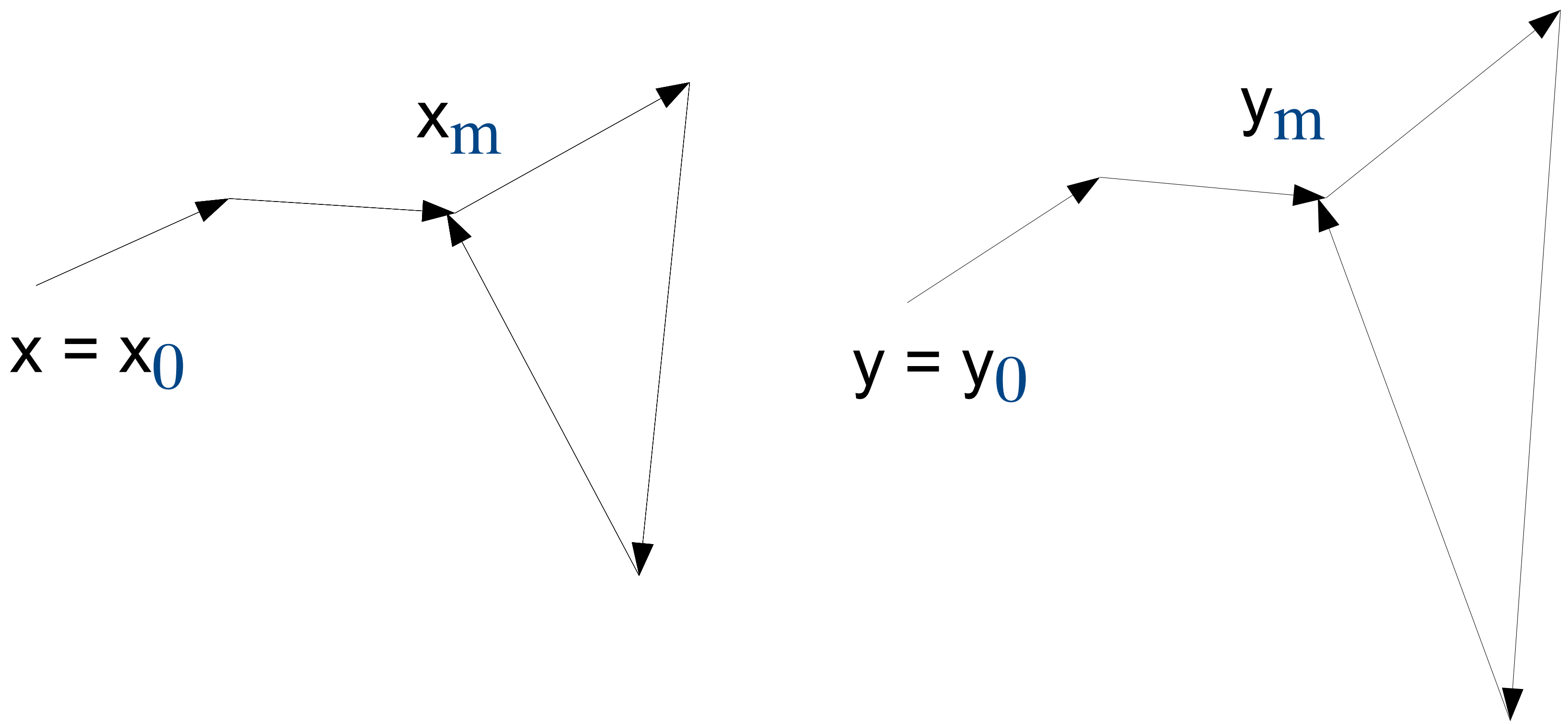
- The variety of L-algebras is *locally finite* : every finitely generated L-algebra is finite
- For every finite k, the free k-generated L-algebra (the *Lindenbaum algebra* of $L\lceil k$) is finite
- Every weak canonical model $M_{L\lceil k}$ is finite.

Proposition Every modal logic of finite modal depth is locally tabular.

Lemma on repeating positions

Let M be a Kripke model, x, y ∈ M. Suppose x $\equiv_n$ y and

moreover, the Duplicator has a winning strategy s in $BG_n(x; y)$ such that every play controlled by s has at least two repeating positions. Then x $\equiv_{n+1}$ y.



Correlation between properties of logics

TABULARITY ⇒ FMD ⇒ LOCAL TABULARITY ⇒ FMP

1. Theorem If F is finite, then $md(L(F)) \leq |F|^2+1$.

Proof: The Pigeonhole principle gives repeating positions.

3. Well-known

2. Easy: there are finitely many k-formulas of bounded modal depth up to equivalence in the basic modal logic.

PROBLEM 1 Does every locally tabular logic have the finite modal depth? (Conjecture:no)

PROBLEM 2 Is there a better upper bound for modal depth of tabular logics? (Conjecture:yes)

Examples of FMD-logics-1

$$md(\mathbf{K} + \Box^n\bot) = n\text{-}1$$

and more generally,

$$md(\mathbf{K}_N + \Box^n\bot) = n\text{-}1$$

where

$$\Box A := \Box_1 A \wedge \ldots \wedge \Box_N A.$$

The axiom $\Box^n\bot$ forbids paths of length n in Kripke frames:

$x_1 R x_2 \ldots R x_n$, where $R = R_1 \cup \ldots \cup R_N$

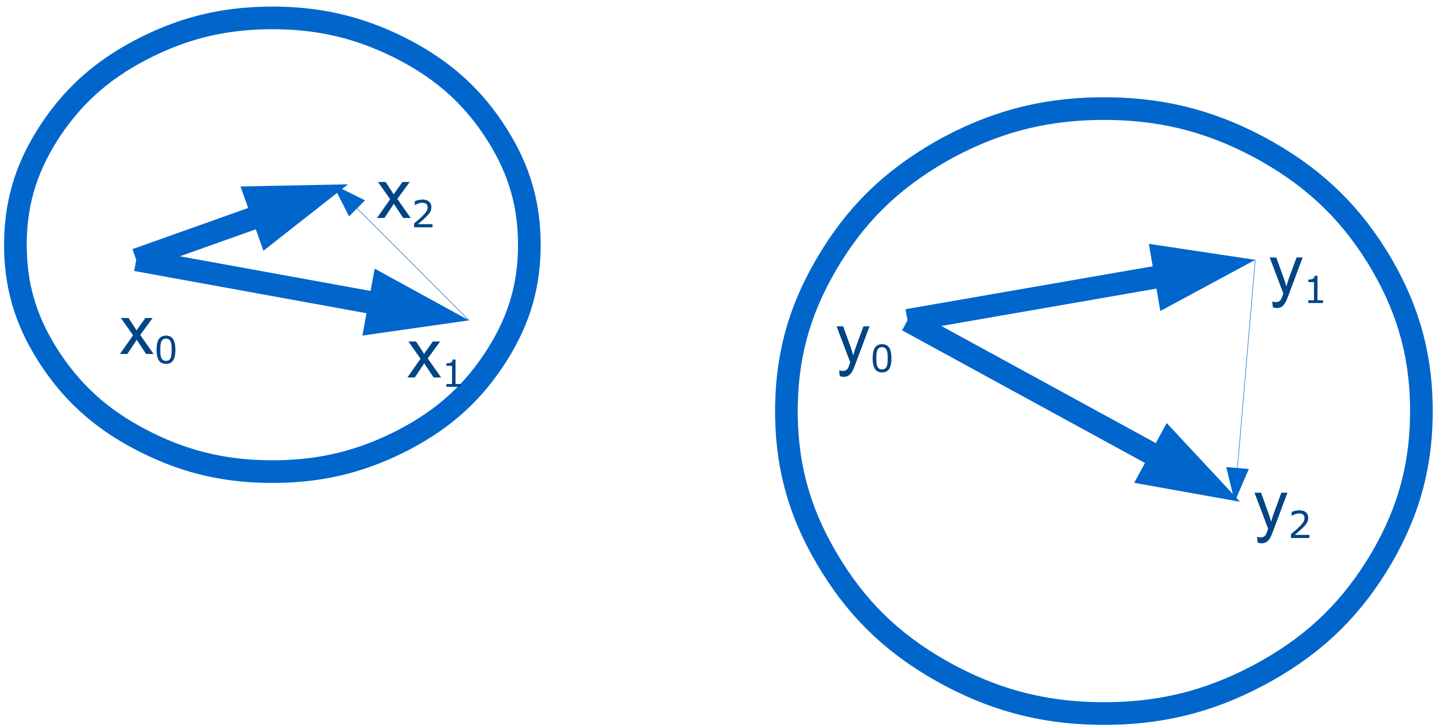
Proof for the upper bound: every play of a bisimulation game contains at most (n-1) rounds.

An earlier result: $\mathbf{K}_N + \Box^n\bot$ is locally tabular (Gabbay & Sh, 1998; a routine proof by induction).

Examples of FMD-logics-2

md(**S5**) = 1 (a well-known fact)

Proof. If Duplicator can win the 1-game, she can win the 2-game



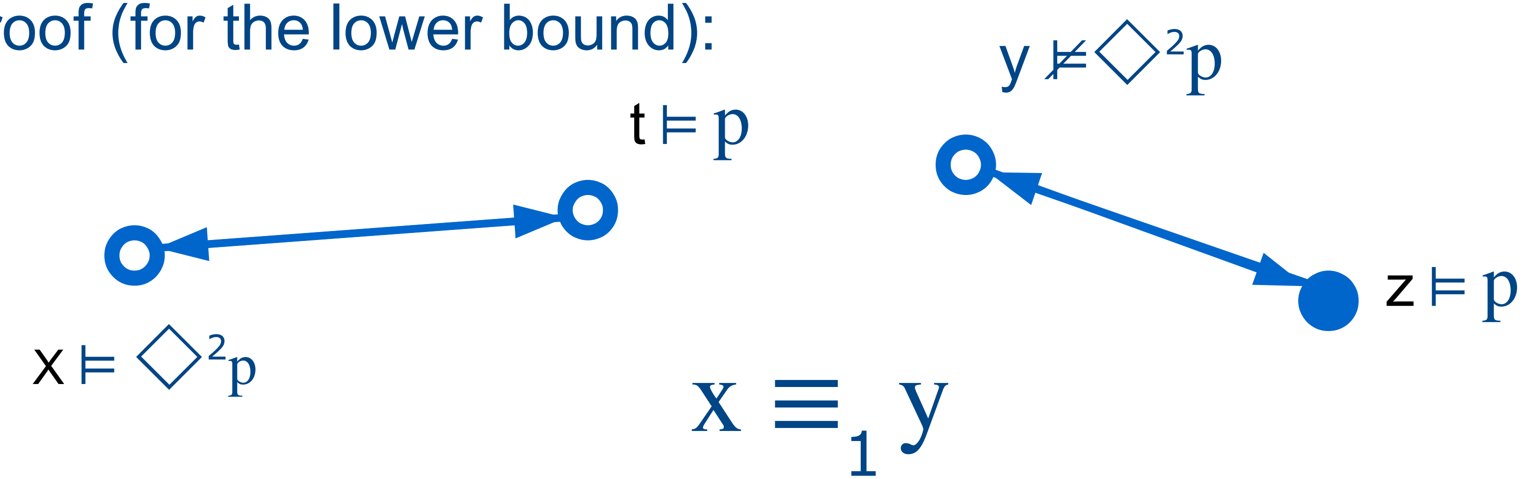
Examples of FMD-logics-3

$md(\mathbf{DL}) = 2$

DL is the *difference logic*

$$\mathbf{DL} = \mathbf{K} + \Diamond\Box p \rightarrow p + \Diamond\Diamond p \rightarrow p \vee \Diamond p$$

- **DL** is complete w.r.t inequality frames $(W, \neq_W)$.
- Arbitary **DL-**frames are obtained from **S5-**frames (equivalence frames) by making some points irreflexive.
- Proof (for the lower bound):



Examples of FMD-logics-4

$md(\mathbf{Grz}+bd_n) \leq 2n-1$

$md(\mathbf{Grz3}+bd_n) = n-1$

Grz is the logic of finite partial orders,

Grz3 is the logic of finite chains

In transitive Kripke frames bd_n forbids *chains of clusters of length n+1* : $x_1Rx_2\ldots Rx_{n+1}$, where $\neg x_iRx_{i+1}$ for each i.

$$bd_n = \neg\Diamond(Q_1 \wedge \Diamond(Q_2 \wedge \ldots \wedge \Diamond Q_{n+1})),$$

$$Q_i = p_i \wedge \bigwedge\{\neg\Diamond p_j \mid 1 \leq j < i\}.$$

Grz3 + bd_n = **L**(n-element chain)

Examples of FMD-logics-5

md(**Grz3**+bd_2) = 1, while 1< md(**Grz**+bd_2) ≤ 3 (probably, =2).

(0,1 show the truth values of p)

Here $x \equiv_1 y$, but $x \not\equiv_2 y$:

Duplicator wins after 1 round.

Spoiler wins after 2 rounds.

Examples of FMD-logics-6

$md(\mathbf{K4}+bd_n) \leq 4n - 3$

Theorem (Segerberg 1971;Maksimova 1975) For L ⊇ **K4**

L is locally tabular iff L is of finite transitive depth.

Def L is of *finite transitive depth* if $L \vdash bd_n$ for some n.

Thus

- Every locally tabular extension of **K4** has the FMD.

PROBLEM Is there a similar criterion for extensions of **K**?

Examples of FMD-logics-7

If md(L) = m, then $md([\mathbf{K}+\Box^n\bot, L]) \leq (m+1)n-1$

Def. The commutative join (commutator)

$[L_1, L_2] := L_1 * L_2$ (the fusion) +

$\blacksquare_j \Box_i p \leftrightarrow \Box_i \blacksquare_j p$ (commutation axioms)

$\blacklozenge_j \Box_i p \to \blacksquare_j \Diamond_i p$ (Church-Rosser axioms)

Tabularity criterion-1

Theorem (Chagrov 1994)

L is tabular iff $L \vdash \alpha_n \wedge Alt_n$ for some n.

The formulas α_n , Alt_n correspond to universal conditions on frames:

- α_n forbids simple paths of length n:

  $x_1Rx_2...Rx_n$, where all the x_i are different.

- Alt_n forbids n-branching: $xRx_1,..., xRx_n$, , where all the x_i are different.

Tabularity criterion-2

$$\alpha_n = \neg\Diamond(P_1 \wedge \Diamond(P_2 \wedge \ldots \Diamond(P_{n-1} \wedge \Diamond P_n)\ldots)),$$

$$Alt_n = \neg(\Diamond P_1 \wedge \Diamond P_2 \wedge \ldots \wedge \Diamond P_n),$$

where

$$P_i = \neg p_i \wedge \bigwedge\{p_j \mid 1 \leq j \leq n,\ j \neq i\}.$$

Theorems on local tabularity

1. Every logic $\mathbf{K}_N + \alpha_n$ (Chagrov's formula) is locally tabular.

Remarks:

- The proof does not give the FMD
- This theorem was conjectured in 1994 by Chagrov.

2. The logics $[\mathbf{K}_N + \alpha_n, \mathbf{K}_{N'} + \Box^n\bot]$, $[\mathbf{K}_N + \alpha_n, \mathbf{S5}]$ are locally tabular.

Remark. In general products and commutative joins do not preserve local tabularity, a counterexample is $[\mathbf{S5},\mathbf{S5}] = \mathbf{S5}^2$ (Tarski).

THANK YOU!

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Logics

- **K** = **L**(all frames)
- **K4** := **K** + $\Diamond\Diamond p \to \Diamond p$ = **L**(all transitive frames)
- **S4** := **K4** + $p \to \Diamond p$ = **L**(all transitive reflexive frames)

  = **L**(all partial orders)

- **Grz** := **S4** + $\neg(p \wedge \Box(p \to \Diamond(\neg p \wedge \Diamond p)))$

  = **L**(all finite partial orders)

- **Grz3** := **Grz** + $\Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p)$

  = **L**(all finite chains)

- **S5** := **S4** + $\Diamond\Box p \to p$ = **L**(all equivalence frames)

  = **L**(all universal frames [clusters])

All these logics have the FMP, so they are decidable.