Complete axiomatizations of lexicographic sums and products of modal logics

Philippe Balbiani[1] Ilya Shapirovsky[2]

[1] Institut de Recherche en Informatique de Toulouse, CNRS — Toulouse University
[2] Institute for Information Transmission Problems, Russian Academy of Sciences

Topology, Algebra, and Categories in Logic (TACL 2015)
Ischia (Italy), June 22, 2015

We consider two natural operations on modal logics —

lexicographic (or *ordered*)
sums and *products*.

For such systems we present general completeness results.

We consider two natural operations on modal logics —

lexicographic (or *ordered*)
sums and *products*.

For such systems we present general completeness results.

Like "usual" product of modal logics, the lexicographic sum and the lexicographic product of modal logics are defined semantically — via corresponding operation on their frames.

Sum of frames

Definition

Let $\mathsf{I} = (I, S)$ be a frame, $\{\mathsf{F}_i = (W_i, R_i) \mid i \in I\}$ be a family of frames. The *lexicographic* (or *ordered*) *sum* $\sum\limits_{\mathsf{I}} \mathsf{F}_i$ is the frame (W, R_+, S_+), where W is the disjoin sum $\sum\limits_{I} W_i = \{(w, i) \mid i \in I, w \in W_i\}$, and

$$(w,i)R_+(u,j) \iff i = j \;\&\; wR_iu,$$
$$(w,i)S_+(u,j) \iff iSj.$$

I is "vertical", F_i are "horizontal".

Sum of logics

Definition

$\sum_{L_2} L_1$ is the logic of sums where "horizontal" frames are L_1-frames, and the "vertical" frame is an L_2-frame:

$$\sum_{L_2} L_1 = \textsc{Log}(\{\sum_{\mathsf{I}} \mathsf{F}_i \mid \mathsf{I} \models L_2,\ \{\mathsf{F}_i \mid i \text{ in } \mathsf{I}\} \models L_1\}).$$

Problem

To construct the axiomatization of $\sum_{L_2} L_1$, knowing the logics L_1, L_2.

Some history

In 2007, Lev Beklemishev constructed the axiomatization of

$$\sum_{GL} GL$$

Some history

In 2007, Lev Beklemishev constructed the axiomatization of

$$\sum_{GL} GL, \quad \sum_{GL}\left(\sum_{GL} GL\right), \quad \sum_{GL}\left(\sum_{GL}\left(\sum_{GL} GL\right)\right), \quad \dots$$

[L. Beklemishev. Kripke semantics for provability logic *GLP*. 2010]

Some history

In 2007, Lev Beklemishev constructed the axiomatization of

$$\sum_{GL} GL, \quad \sum_{GL}\left(\sum_{GL} GL\right), \quad \sum_{GL}\left(\sum_{GL}\left(\sum_{GL} GL\right)\right), \quad \dots$$

[L. Beklemishev. Kripke semantics for provability logic *GLP*. 2010]

In the context of decidability and complexity, the sum operation turns out to be a good operation!

In many cases the sum operation preserves complexity of logics. In particular, all the above logics are in *PSPACE* ([Sh, 2008]); it follows that *GLP* is in *PSPACE*.

Simultaneously, Sergey Babenyshev and Vladimir Rybakov developed filtrations for sums, and proved a number of decidability results.
[Babenyshev, Rybakov. Logics of Kripke meta-models. 2010]

$$\alpha = \Box_2 p \to \Box_1 \Box_2 p, \quad \beta = \Box_2 p \to \Box_2 \Box_1 p, \quad \gamma = \Diamond_2 p \to \Box_1 \Diamond_2 p$$

$$\sum_{GL} GL = GL * GL + \{\alpha, \beta, \gamma\}$$

$$\alpha = \Box_2 p \to \Box_1 \Box_2 p, \quad \beta = \Box_2 p \to \Box_2 \Box_1 p, \quad \gamma = \Diamond_2 p \to \Box_1 \Diamond_2 p$$

$$\sum_{GL} GL = GL * GL + \{\alpha, \beta, \gamma\}$$

What is the meaning of the formulas α, β, γ?

$$\alpha = \Box_2 p \to \Box_1 \Box_2 p, \quad \beta = \Box_2 p \to \Box_2 \Box_1 p, \quad \gamma = \Diamond_2 p \to \Box_1 \Diamond_2 p$$

$$\sum_{GL} GL = GL * GL + \{\alpha, \beta, \gamma\}$$

What is the meaning of the formulas α, β, γ?

α, β, γ are Sahlqvist formulas. For $\mathsf{F} = (W, R_1, R_2)$, we have:

$$\mathsf{F} \models \alpha \iff R_1 \circ R_2 \subseteq R_2;$$
$$\mathsf{F} \models \beta \iff R_2 \circ R_1 \subseteq R_2;$$
$$\mathsf{F} \models \gamma \iff R_1^{-1} \circ R_2 \subseteq R_2.$$

$$\alpha = \Box_2 p \to \Box_1 \Box_2 p, \quad \beta = \Box_2 p \to \Box_2 \Box_1 p, \quad \gamma = \Diamond_2 p \to \Box_1 \Diamond_2 p$$

$$\sum_{GL} GL = GL * GL + \{\alpha, \beta, \gamma\}$$

What is the meaning of the formulas α, β, γ?

α, β, γ are Sahlqvist formulas. For $\mathsf{F} = (W, R_1, R_2)$, we have:

$$\mathsf{F} \models \alpha \iff R_1 \circ R_2 \subseteq R_2;$$
$$\mathsf{F} \models \beta \iff R_2 \circ R_1 \subseteq R_2;$$
$$\mathsf{F} \models \gamma \iff R_1^{-1} \circ R_2 \subseteq R_2.$$

Lemma (2014)

Consider a rooted frame $\mathsf{F} = (W, R_1, R_2)$.
$\mathsf{F} \models \alpha \land \beta \land \gamma$ iff F is a p-morphic image of a sum.

$$\alpha = \Box_2 p \to \Box_1 \Box_2 p, \quad \beta = \Box_2 p \to \Box_2 \Box_1 p, \quad \gamma = \Diamond_2 p \to \Box_1 \Diamond_2 p$$

$$\sum_{GL} GL = GL * GL + \{\alpha, \beta, \gamma\}$$

What is the meaning of the formulas α, β, γ?

α, β, γ are Sahlqvist formulas. For $\mathsf{F} = (W, R_1, R_2)$, we have:

$$\begin{aligned} \mathsf{F} \models \alpha &\iff R_1 \circ R_2 \subseteq R_2; \\ \mathsf{F} \models \beta &\iff R_2 \circ R_1 \subseteq R_2; \\ \mathsf{F} \models \gamma &\iff R_1^{-1} \circ R_2 \subseteq R_2. \end{aligned}$$

Lemma (2014)

Consider a rooted frame $\mathsf{F} = (W, R_1, R_2)$.
$\mathsf{F} \models \alpha \wedge \beta \wedge \gamma$ *iff* F *is a* p*-morphic image of a sum.*

Corollary

$$\sum_{K} K = K * K + \{\alpha, \beta, \gamma\}$$

By a *closed sentence* we mean the standard translation of a closed modal formula.

Horn sentences: $\forall x_1 \dots x_n(\psi_1 \wedge \dots \wedge \psi_k \to \psi_0)$, where ψ_i are atoms.

A logic L is *Horn axiomatizable*, if $\mathrm{Frames}(L)$ is an elementary class that is defined by Horn sentences and closed sentences. The standard systems $\mathbf{K}, \mathbf{T}, \mathbf{B}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}, \dots$ are examples of Horn axiomatizable logics.

Theorem 1

Let $L_1 * L_2 + \{\alpha, \beta, \gamma\}$ be Kripke complete, L_2 Horn axiomatizable. Then $\sum_{L_2} L_1 = L_1 * L_2 + \{\alpha, \beta, \gamma\}$.

Corollary

Let L_1 and L_2 be canonical unimodal logics, L_2 Horn axiomatizable. Then $\sum_{L_2} L_1 = L_1 * L_2 + \{\alpha, \beta, \gamma\}$.

Lexicographic products of frames

Definition

Let $\mathsf{I} = (I, S)$ be a frame, $\{\mathsf{F}_i = (W_i, R_i) \mid i \in I\}$ be a family of frames. The *lexicographic* (or *ordered*) *sum* $\sum_{\mathsf{I}} \mathsf{F}_i$ is the frame (W, R_+, S_+), where W is the disjoin sum $\sum_I W_i = \{(w, i) \mid i \in I, w \in W_i\}$, and

$$
\begin{aligned}
(w, i) R_+ (u, j) &\iff i = j \ \& \ w R_i u, \\
(w, i) S_+ (u, j) &\iff i S j.
\end{aligned}
$$

Lexicographic products of frames

Definition

Let $\mathsf{I} = (I, S)$ be a frame, $\{\mathsf{F}_i = (W_i, R_i) \mid i \in I\}$ be a family of frames. The *lexicographic* (or *ordered*) *sum* $\sum\limits_{\mathsf{I}} \mathsf{F}_i$ is the frame (W, R_+, S_+), where W is the disjoin sum $\sum\limits_{I} W_i = \{(w, i) \mid i \in I, w \in W_i\}$, and

$$\begin{aligned}(w,i)R_+(u,j) &\iff i = j \ \&\ wR_iu,\\ (w,i)S_+(u,j) &\iff iSj.\end{aligned}$$

If for all i $\mathsf{F}_i = \mathsf{F}$, we write $\mathsf{F} \leftthreetimes \mathsf{I}$ for $\sum\limits_{\mathsf{I}} \mathsf{F}_i$; the frame $\mathsf{F} \leftthreetimes \mathsf{I}$ is called the *lexicographic product of frames* F *and* I.

Lexicographic products of logics

Definition

For logics L_1, L_2, put

$$L_1 \leftthreetimes L_2 = \textsc{Log}(\{\mathsf{F} \leftthreetimes \mathsf{I} \mid \mathsf{F} \models L_1, \mathsf{I} \models L_2\}).$$

Problem

To construct the axiomatization of $L_1 \leftthreetimes L_2$, knowing the logics L_1, L_2.

Lexicographic products of logics

Definition

For logics L_1, L_2, put

$$L_1 \leftthreetimes L_2 = \textsc{Log}(\{\mathsf{F} \leftthreetimes \mathsf{I} \mid \mathsf{F} \models L_1, \mathsf{I} \models L_2\}).$$

Problem

To construct the axiomatization of $L_1 \leftthreetimes L_2$, knowing the logics L_1, L_2.

[Ph. Balbiani, Axiomatization and completeness of lexicographic products of modal logics. 2009.]

Theorem 2 (2009; 2014)

If

- L_1 and L_2 are Horn axiomatizable Kripke complete logics,
- $\Diamond\top \in L_1$,

then

$$L_1 \leftthreetimes L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\},$$

Theorem 2 (2009; 2014)

If

- L_1 and L_2 are Horn axiomatizable Kripke complete logics,
- $\Diamond\top \in L_1$,

then

$$L_1 \leftthreetimes L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\},$$

and hence,

$$L_1 \leftthreetimes L_2 = \sum_{L_2} L_1 .$$

Theorem 2 (2009; 2014)

If

- L_1 and L_2 are Horn axiomatizable Kripke complete logics,
- $\Diamond\top \in L_1$,

then

$$L_1 \leftthreetimes L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\},$$

and hence,

$$L_1 \leftthreetimes L_2 = \sum_{L_2} L_1 .$$

Question (2009)

$$\mathsf{K} \leftthreetimes \mathsf{K} = ?$$

Φ is the set of all closed formulas in the modal language $ML(\Box_1)$.

Theorem 3

If L_1 and L_2 are Horn axiomatizable Kripke complete logics, then

$$L_1 \times L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\} \cup \Xi_1 \cup \Xi_2 \cup \Xi_3,$$

where

$$\begin{aligned} \Xi_1 &= \{\Diamond_2\Diamond_2 p \wedge \Diamond_2\varphi \to \Diamond_2(\Diamond_2 p \wedge \varphi) \mid \varphi \in \Phi\}, \\ \Xi_2 &= \{\Diamond_2\Box_2\bot \wedge \Diamond_2\varphi \to \Diamond_2(\Box_2\bot \wedge \varphi) \mid \varphi \in \Phi\}, \\ \Xi_3 &= \{\Diamond_2^i\varphi \to \Box_2^j(\Diamond_2\top \to \Diamond_2\varphi) \mid i, j \geq 0,\ \varphi \in \Phi\}. \end{aligned}$$

Note that

if $\Diamond\top \in L_1$, then

$$L_1 * L_2 + \{\alpha, \beta, \gamma\} \cup \Xi_1 \cup \Xi_2 \cup \Xi_3 = L_1 * L_2 + \{\alpha, \beta, \gamma\}$$

By the way...

Similar situation appears in topological (neighborhood) products of modal logics:

[J. van Benthem, G Bezhanishvili, B. ten Cate, D. Sarenac, 2006], [Kudinov, 2012]

$\mathbf{S4} \times_N \mathbf{S4} = \mathbf{S4} * \mathbf{S4}$,
$(\mathbf{K} + \Diamond\top) \times_N \mathbf{S4} = (\mathbf{K} + \Diamond\top) * \mathbf{S4}$,
$(\mathbf{K} + \Diamond\top) \times_N (\mathbf{K} + \Diamond\top) = (\mathbf{K} + \Diamond\top) * (\mathbf{K} + \Diamond\top)$,
...

By the way...

Similar situation appears in topological (neighborhood) products of modal logics:

[J. van Benthem, G Bezhanishvili, B. ten Cate, D. Sarenac, 2006], [Kudinov, 2012]

$\mathbf{S4} \times_N \mathbf{S4} = \mathbf{S4} * \mathbf{S4}$,
$(\mathsf{K} + \Diamond\top) \times_N \mathbf{S4} = (\mathsf{K} + \Diamond\top) * \mathbf{S4}$,
$(\mathsf{K} + \Diamond\top) \times_N (\mathsf{K} + \Diamond\top) = (\mathsf{K} + \Diamond\top) * (\mathsf{K} + \Diamond\top)$,
...

[Kudinov, 2014]

$$\mathsf{K} \times_N \mathsf{K} = \mathsf{K} * \mathsf{K} + \Delta,$$

where

$$\Delta = \{\phi \to \Box_2\phi \mid \phi \text{ is closed } \Box_1\text{-formula}\} \cup \{\psi \to \Box_1\psi \mid \psi \text{ is closed } \Box_2\text{-formula}\}.$$

Φ is the set of all closed formulas in the modal language $ML(\Box_1)$.

Theorem 3

If L_1 and L_2 are Horn axiomatizable Kripke complete logics, then

$$L_1 \times L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\} \cup \Xi_1 \cup \Xi_2 \cup \Xi_3,$$

where

$$\begin{aligned}
\Xi_1 &= \{\Diamond_2\Diamond_2 p \wedge \Diamond_2\varphi \to \Diamond_2(\Diamond_2 p \wedge \varphi) \mid \varphi \in \Phi\},\\
\Xi_2 &= \{\Diamond_2\Box_2\bot \wedge \Diamond_2\varphi \to \Diamond_2(\Box_2\bot \wedge \varphi) \mid \varphi \in \Phi\},\\
\Xi_3 &= \{\Diamond_2^i\varphi \to \Box_2^j(\Diamond_2\top \to \Diamond_2\varphi) \mid i,j \geq 0,\ \varphi \in \Phi\}.
\end{aligned}$$

Note that

if $\Diamond\top \in L_1$, then

$$L_1 * L_2 + \{\alpha, \beta, \gamma\} \cup \Xi_1 \cup \Xi_2 \cup \Xi_3 = L_1 * L_2 + \{\alpha, \beta, \gamma\}$$

Decidability and complexity of lexicographic products

From the computational point of view, lexicographic products are safer than "usual" modal products.

For example, the satisfiability problem for $\mathbf{S4} \leftthreetimes \mathbf{S4}$ is in *PSPACE*.

Theorem

Let L_1, L_2 be Kripke complete unimodal logics, and both L_1 and L_2 admit filtration. Then L_1 and L_2 have the $\leftthreetimes$-fmp, i.e.,

$$L_1 \leftthreetimes L_2 = \textsc{Log}(\{\mathsf{F}_1 \leftthreetimes \mathsf{F}_2 \mid \mathsf{F}_i \models L_i,\ \mathsf{F}_i \text{ are finite}\}).$$

Thank you!