

Relational lattices via duality

Luigi Santocanale*

*LIF, Aix-Marseille Université

E-mail: luigi.santocanale@lif.univ-mrs.fr

URL: <http://pageperso.lif.univ-mrs.fr/~luigi.santocanale>

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Some works I'm in debt to . . .

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Databases, tables, sqls ...

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The screenshot shows the Amazon.fr homepage. At the top, there's a navigation bar with the Amazon logo, search bar, and various account links. A large banner features a Kindle Paperwhite advertisement with the text "NOUVEAU kindle paperwhite À partir de 129,99€" and "Notre liseuse phare, encore améliorée". Below this, another banner for "kindle voyage À partir de 189,99€" is visible. The main content area is titled "Recommandations pour vous en Livres" and displays a row of book covers including "Malden's Orange", "Catch-22", "Bird of the Day", "RAY BRADBURY Fahrenheit 451", "the CATCHER in the RYE", and "BOAT STRENGTH". To the right, there's a section for a Dyson V6 Absolute vacuum cleaner with a 4.5-star rating and a price of EUR 386,19. The browser's address bar shows the URL "www.amazon.fr" and the taskbar at the bottom displays several open files.

Databases, tables, sqls ...

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The screenshot shows the phpMyAdmin interface in a browser window. The address bar shows the URL: `https://[redacted]phpmyadmin/index.php?lang=en-iso-8`. The interface displays the table structure for the 'hostadm' database. The table has 10 columns: otype, fname, sequence, type, capture, tsize, values, mandatory, unique, and default. The rows represent different device types and their associated fields.

	otype	fname	sequence	type	capture	tsize	values	mandatory	unique	default
<input type="checkbox"/>	Miso Device	Name	0	oname	text	40		Y	Y	
<input type="checkbox"/>	Miso Device	Description	10	string	text	80		N	N	
<input type="checkbox"/>	Miso Device	Serial Number	20	string	text	40		N	N	
<input type="checkbox"/>	Miso Device	Computer Room	22	string	radio	NULL	Pathfoot,Cottrell,Library,Library Ante-Room,NonCR	N	N	
<input type="checkbox"/>	Miso Device	Rack	23	string	text	10	NULL	N	N	
<input type="checkbox"/>	Miso Device	Rack Position	24	number	text	3	NULL	N	N	
<input type="checkbox"/>	Miso Device	Height (U)	27	number	text	3	NULL	N	N	
<input type="checkbox"/>	Miso Device	Data Port	30	objectlist				N	N	
<input type="checkbox"/>	Miso Device	Power Supply	40	objectlist				N	N	
<input type="checkbox"/>	Miso Device	Notes	80	string	textbox	40*4		N	N	

At the bottom of the interface, there are controls for 'Show: 30 row(s) starting from record # 0', 'in horizontal mode and repeat headers after 100 cells', and buttons for 'Insert new row', 'Print view', 'Print view (with full texts)', and 'Export'.

Databases, tables, sqls ...

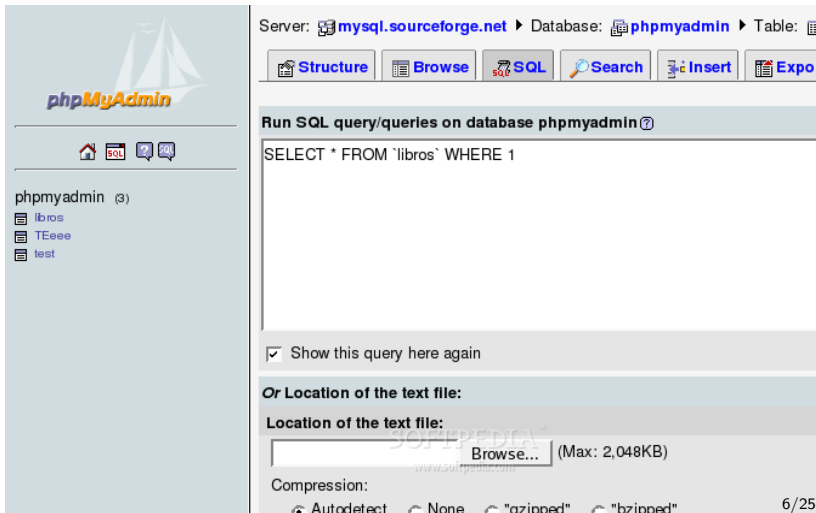
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The screenshot shows the phpMyAdmin web interface. At the top, the server is identified as `mysql.sourceforge.net`, the database as `phpmyadmin`, and the table as `libros`. A toolbar contains buttons for `Structure`, `Browse`, `SQL`, `Search`, `Insert`, and `Export`. The main area displays the query `SELECT * FROM `libros` WHERE 1` and its result. Below the query, there is a checkbox for `Show this query here again` which is checked. The section `Or Location of the text file:` includes a field for the file location with a `Browse...` button and a size limit of `(Max: 2,048KB)`. A `Compression:` section is partially visible at the bottom, showing options like `Autodetect`, `None`, `"gzipped"`, and `"bzipped"`.

Server: `mysql.sourceforge.net` Database: `phpmyadmin` Table: `libros`

Structure Browse SQL Search Insert Export

Run SQL query/queries on database phpmyadmin ?

```
SELECT * FROM `libros` WHERE 1
```

Show this query here again

Or Location of the text file:

Location of the text file: Browse... (Max: 2,048KB)

Compression:

Autodetect None "gzipped" "bzipped"

Mixing up tables: the outer join

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Name	Surname	Item
Luigi	Santocanale	33
Alan	Turing	21

\wedge

Item	Description
33	Book
33	Livre
21	Machine

=

Name	Surname	Item	Description
Luigi	Santocanale	33	Book
Luigi	Santocanale	33	Livre
Alan	Turing	21	Machine

Mixing up tables: the inner union

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Name	Surname	Item
Luigi	Santocanale	33
Alan	Turing	21

\vee

Name	Surname	Sport
Diego	Maradona	Football
Usain	Bolt	Athletics

=

Name	Surname
Luigi	Santocanale
Alan	Turing
Diego	Maradona
Usain	Bolt

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Saving the world with lattice theory and logic

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Proposition (Spight & Tropashko [3])

The set of tables, whose columns are indexed by a subset of A and values are from a set D , is a lattice, with external join as meet and inner union as join.

Saving the world with lattice theory and logic

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Proposition (Spight & Tropashko [3])

The set of tables, whose columns are indexed by a subset of A and values are from a set D , is a lattice, with external join as meet and inner union as join.

Goals

- Study the equational theory of relational lattices.
- Use knowledge of the equational theory to improve database queries.

Saving the world with lattice theory and logic

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Proposition (Spight & Tropashko [3])

The set of tables, whose columns are indexed by a subset of A and values are from a set D , is a lattice, with external join as meet and inner union as join.

Goals

- Study the equational theory of relational lattices.
- Use knowledge of the equational theory to improve database queries.
- Get a job with Oracle,
- ... a house on the sea in California, ...

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The lattice $R(D, A)$

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A a set of attributes, D a set of values.

A member of $R(D, A)$ is a pair (X, T) with $X \subseteq A$ and $T \subseteq D^X$.

We have

$$(X_1, T_1) \leq (X_2, T_2) \text{ iff } X_2 \subseteq X_1 \text{ and } T_1|_{X_2} \subseteq T_2.$$

A bit of categories ...

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$R(D, A)$ is the category of elements of the functor

$$\begin{array}{ccccc} P(A)^{op} & \rightarrow & \text{Set} & \rightarrow & \mathcal{SL}_V \\ X & \mapsto & D^X & \mapsto & P(D^X). \end{array}$$

The image of a pullback square satisfies the Beck-Chevalley property:

$$\begin{array}{ccc} P(D^{X_1 \cap X_2}) & \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{j} \end{array} & P(D^{X_2}) \\ \begin{array}{c} \uparrow j \\ \downarrow i \end{array} & \begin{array}{c} \nearrow \text{dotted} \\ \searrow \text{dotted} \end{array} & \begin{array}{c} \uparrow j \\ \downarrow i \end{array} \\ P(D^{X_1}) & \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{j} \end{array} & P(D^{X_3}) \end{array}$$

A bit of algebra . . .

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We have an action

$$j : P(A) \longrightarrow \text{Clops}(P(D^A))$$

giving rise to a semidirect product construction:

$$\begin{aligned} R(D, A) &\equiv P(A) \ltimes_j P(D^A) \\ &:= \{(X, j_X(T)) \mid X \in P(A), T \in D^A\}. \end{aligned}$$

This action satisfies the BC-Malcev-property:

$$j_{X \cup Y} = j_X \circ j_Y.$$

$R(D, A)$ from a closure operator

Relational
lattices

Define an ultrametric distance on D^A with values in $P(A)$:

$$\delta(f, g) = \{x \in A \mid f(x) \neq g(x)\}.$$

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$R(D, A)$ from a closure operator

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Define an ultrametric distance on D^A with values in $P(A)$:

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This distance is

- 1 symmetric: $\delta(f, g) = \delta(g, f)$,
- 2 it has the Beck-Chevalley-Malcev property: *if $\delta(f, g) \subseteq A \cup B$, then there exists h such that $\delta(f, h) \subseteq A$ and $\delta(h, g) \subseteq B$.*

$R(D, A)$ from a closure operator

Relational
lattices

Define an ultrametric distance on D^A with values in $P(A)$:

$$\delta(f, g) = \{x \in A \mid f(x) \neq g(x)\}.$$

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This distance is

- 1 symmetric: $\delta(f, g) = \delta(g, f)$,
- 2 it has the Beck-Chevalley-Malcev property: if $\delta(f, g) \subseteq A \cup B$, then there exists h such that $\delta(f, h) \subseteq A$ and $\delta(h, g) \subseteq B$.

A subset X of $A + D^A$ is *closed* if $\delta(f, g) \cup \{g\} \subseteq X$ implies $f \in X$.

Proposition (Litak et al. [1])

$R(D, A)$ is isomorphic to the lattice of closed subsets of $A + D^A$.

OD-graph based duality

Relational
lattices

Given a finite lattice L , its OD-graph is the structure $(J(L), \leq, \triangleleft)$, with

- $(J(L), \leq)$: ordered join-irreducible els.,
- $j \triangleleft C$ iff $C \subseteq J(L)$ and C is a minimal join-cover of j .

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Given a finite lattice L , its OD-graph is the structure $(J(L), \leq, \triangleleft)$, with

- $(J(L), \leq)$: ordered join-irreducible els.,
- $j \triangleleft C$ iff $C \subseteq J(L)$ and C is a minimal join-cover of j .

In particular:

$$j \leq \bigvee X \quad \text{iff}$$

there exists $C \subseteq J(L)$ with $j \triangleleft C$ and $C \ll X$,

OD-graph based duality

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Given a finite lattice L , its OD-graph is the structure $(J(L), \leq, \triangleleft)$, with

- $(J(L), \leq)$: ordered join-irreducible els.,
- $j \triangleleft C$ iff $C \subseteq J(L)$ and C is a minimal join-cover of j .

In particular:

$$j \leq \bigvee X \quad \text{iff} \\ \text{there exists } C \subseteq J(L) \text{ with } j \triangleleft C \text{ and } C \ll X,$$

where

$$X \ll Y \text{ iff } \forall x \in X \exists y \in Y \quad \text{s.t.} \quad x \leq y.$$

Use of duality

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- semantics, game semantics, ...
- validity of equations ...
- counter-model construction ...
- correspondence results ...
- heuristics ...

Use of duality

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Axiomatizations

- semantics, game semantics, ...
- validity of equations ...
- counter-model construction ...
- correspondence results ...
- heuristics ...

The lattices $R(D, A)$ might not be finite, but they are
more-than-perfect

That is: they enjoy the useful properties of the finite ones.

Minimal join-covers in $R(D, A)$

Relational
lattices

$R(D, A)$ is an atomistic lattice: its atoms are of the form

- \hat{a} , for $a \in A$ (these are join-prime);
- \hat{f} , for $f \in D^A$.

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Minimal join-covers in $R(D, A)$

Relational
lattices

$R(D, A)$ is an atomistic lattice: its atoms are of the form

- \hat{a} , for $a \in A$ (these are join-prime);
- \hat{f} , for $f \in D^A$.

(Possibly infinite) minimal join-covers are those of the form

$$\hat{f} \leq \bigvee_{a \in \delta(f,g)} \hat{a} \vee \hat{g}$$

for each $g \in D^A$.

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$R(D, A)$ is an atomistic lattice: its atoms are of the form

- \hat{a} , for $a \in A$ (these are join-prime);
- \hat{f} , for $f \in D^A$.

(Possibly infinite) minimal join-covers are those of the form

$$\hat{f} \leq \bigvee_{a \in \delta(f,g)} \hat{a} \vee \hat{g}$$

for each $g \in D^A$.

Remarkable property:

Each minimal join-cover has at most one non-join-prime element.

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Known equations [1]: AxRL1 and AxRL2

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AxRL1 is:

$$x \wedge ((y \wedge (z \vee x)) \vee (z \wedge (y \vee x))) \leq (x \wedge y) \vee (x \wedge z)$$

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Known equations [1]: AxRL1 and AxRL2

Relational
lattices

AxRL1 is:

$$x \wedge ((y \wedge (z \vee x)) \vee (z \wedge (y \vee x))) \leq (x \wedge y) \vee (x \wedge z)$$

For $u \in \{y, z\}$, set

$$d_\ell^o(u) := (u_0 \vee u_1) \wedge (u_0 \vee u_2), \quad d_\rho^o(u) := u_0 \vee (u_1 \wedge u_2).$$

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Known equations [1]: AxRL1 and AxRL2

Relational
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AxRL1 is:

$$x \wedge ((y \wedge (z \vee x)) \vee (z \wedge (y \vee x))) \leq (x \wedge y) \vee (x \wedge z)$$

For $u \in \{y, z\}$, set

$$d_\ell^o(u) := (u_0 \vee u_1) \wedge (u_0 \vee u_2), \quad d_\rho^o(u) := u_0 \vee (u_1 \wedge u_2).$$

AxRL2 is:

$$\begin{aligned} & x \wedge (d_\ell^o(y) \vee d_\ell^o(z)) \\ & \leq (x \wedge (d_\rho^o(y) \vee d_\rho^o(z))) \vee (x \wedge (d_\ell^o(y) \vee d_\rho^o(z))) \end{aligned}$$

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Known equations [1]: AxRL1 and AxRL2

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AxRL1 is:

$$x \wedge ((y \wedge (z \vee x)) \vee (z \wedge (y \vee x))) \leq (x \wedge y) \vee (x \wedge z)$$

For $u \in \{y, z\}$, set

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AxRL2 is:

$$\begin{aligned} x \wedge (d_\ell^o(y) \vee d_\ell^o(z)) \\ \leq (x \wedge (d_\rho^o(y) \vee d_\rho^o(z))) \vee (x \wedge (d_\ell^o(y) \vee d_\rho^o(z))) \end{aligned}$$

Easy proofs that $R(D, A)$ satisfies these equations using the dual structure.

New equations: UNJP

Set

$$d_\ell(u) := u_0 \wedge (u_1 \vee u_2), \quad d_\rho(u) := (u_0 \wedge u_1) \vee (u_0 \wedge u_2).$$

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New equations: UNJP

Set

$$\mathbf{d}_\ell(u) := u_0 \wedge (u_1 \vee u_2), \quad \mathbf{d}_\rho(u) := (u_0 \wedge u_1) \vee (u_0 \wedge u_2).$$

UNJP is:

$$\begin{aligned} & x \wedge (\mathbf{d}_\ell(y) \vee \mathbf{d}_\ell(z) \vee w) \\ & \leq (x \wedge (\mathbf{d}_\rho(y) \vee \mathbf{d}_\ell(z) \vee w)) \vee (x \wedge (\mathbf{d}_\ell(y) \vee \mathbf{d}_\rho(z) \vee w)). \end{aligned}$$

New equations: UNJP

Relational
lattices

Set

$$d_\ell(u) := u_0 \wedge (u_1 \vee u_2), \quad d_\rho(u) := (u_0 \wedge u_1) \vee (u_0 \wedge u_2).$$

UNJP is:

$$\begin{aligned} & x \wedge (d_\ell(y) \vee d_\ell(z) \vee w) \\ & \leq (x \wedge (d_\rho(y) \vee d_\ell(z) \vee w)) \vee (x \wedge (d_\ell(y) \vee d_\rho(z) \vee w)). \end{aligned}$$

Theorem

UNJP holds in a more-than-perfect lattice iff every minimal join-cover contains at most one non-join-prime element.

Proposition

AXRL2 is derivable from UNJP, but not the converse.

(Throw Mace4, Prover9, and Waldemeister in the trash ...)

Proposition

AXRL2 is derivable from UNJP, but not the converse.

(Throw Mace4, Prover9, and Waldemeister in the trash ...)

Proposition

Can derive from UNJP

$$\begin{aligned} & (x \wedge (t_\ell(y) \vee s_\ell(z) \vee w)) \vee (x \wedge (t_\rho(y) \vee s_\rho(z) \vee w)) \\ &= (x \wedge (t_\rho(y) \vee s_\ell(z) \vee w)) \vee (x \wedge (t_\ell(y) \vee s_\rho(z) \vee w)) \end{aligned}$$

whenever $t_\ell(y) = t_\rho(y)$ and $s_\ell(z) = s_\rho(z)$ hold on distributive lattices.

Other equations: symmetry and the BC property

Relational
lattices

$$x \wedge (y \vee z) \leq (x \wedge (y \vee (z \wedge (x \vee y)))) \vee (x \wedge (z \vee (y \wedge (x \vee z)))) \quad (\text{SymBC}_1)$$

$$x \wedge ((y \wedge z) \vee (y \wedge x) \vee (z \wedge x)) \leq (x \wedge y) \vee (x \wedge z) \quad (\text{Var-AxRL1})$$

$$x \wedge ((x \wedge y) \vee \mathbf{d}_\ell(z)) \leq (x \wedge ((x \wedge y) \vee \mathbf{d}_\rho(z))) \vee (x \wedge \mathbf{d}_\ell(z)) \quad (\text{R-Mod})$$

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Other equations: symmetry and the BC property

Relational
lattices

$$x \wedge (y \vee z) \leq (x \wedge (y \vee (z \wedge (x \vee y)))) \vee (x \wedge (z \vee (y \wedge (x \vee z)))) \quad (\text{SymBC}_1)$$

$$x \wedge ((y \wedge z) \vee (y \wedge x) \vee (z \wedge x)) \leq (x \wedge y) \vee (x \wedge z) \quad (\text{Var-AxRL1})$$

$$x \wedge ((x \wedge y) \vee d_\ell(z)) \leq (x \wedge ((x \wedge y) \vee d_\rho(z))) \vee (x \wedge d_\ell(z)) \quad (\text{R-Mod})$$

Proposition

$UNJP, \text{SymBC}_1, \text{Var-AxRL1}, \text{R-Mod} \vdash \text{AxRL1}.$

Theorem

Assume UNJP. A more-than-perfect lattice satisfies SymBC_1 , Var-AxRL1 , $R\text{-Mod}$ if and only if it is symmetric and satisfies the Beck-Chevalley property.

Theorem

Assume UNJP. A more-than-perfect lattice satisfies SymBC_1 , Var-AxRL1 , $R\text{-Mod}$ if and only if it is symmetric and satisfies the Beck-Chevalley property.

- A *more-than-perfect* lattice in the variety UNJP is symmetric iff ...

Theorem

Assume UNJP. A more-than-perfect lattice satisfies SymBC_1 , Var-AxRL1 , $R\text{-Mod}$ if and only if it is symmetric and satisfies the Beck-Chevalley property.

- A *more-than-perfect* lattice in the variety UNJP is symmetric iff ...
- A *more-than-perfect* lattice in the variety the Beck-Chevalley property iff ...

Towards a completeness theorem?

We can obtain similar lattices from (generalized) ultrametric spaces with distance valued on $P(A)$.

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Towards a completeness theorem?

We can obtain similar lattices from (generalized) ultrametric spaces with distance valued on $P(A)$.

Open problem

Is the above axiomatization complete, w.r.t.

- relational lattices?
- lattices constructed out of ultrametric spaces?

Towards a completeness theorem?

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We can obtain similar lattices from (generalized) ultrametric spaces with distance valued on $P(A)$.

Open problem

Is the above axiomatization complete, w.r.t.

- relational lattices?
- lattices constructed out of ultrametric spaces?

Tentative answer. No, we miss the distance property:

for each join-irreducible elements j, k there exists at most one minimal join-covering $j \triangleleft C$ such that $k \in C$.

Thanks four your attention ...



... $R(D, A)$ s, get me there !!!