Relational lattices via duality

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TACL, Ischia, June 2015

Outline

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Some works I'm in debt to . . .

Tadeusz Litak, Szabolcs Mikulás, and Jan Hidders.
Relational lattices.
In Peter Höfner and et al., editors, *RAMiCS 2014*, volume 8428 of *LNCS*, pages 327–343. Springer, 2014.

Marshall Spight and Vadim Tropashko.
First steps in relational lattice.
CoRR, abs/cs/0603044, 2006.

Marshall Spight and Vadim Tropashko.
Relational lattice axioms.
CoRR, abs/0807.3795, 2008.

Vadim Tropashko.
Relational lattice foundation for algebraic logic.
CoRR, abs/0902.3532, 2009.

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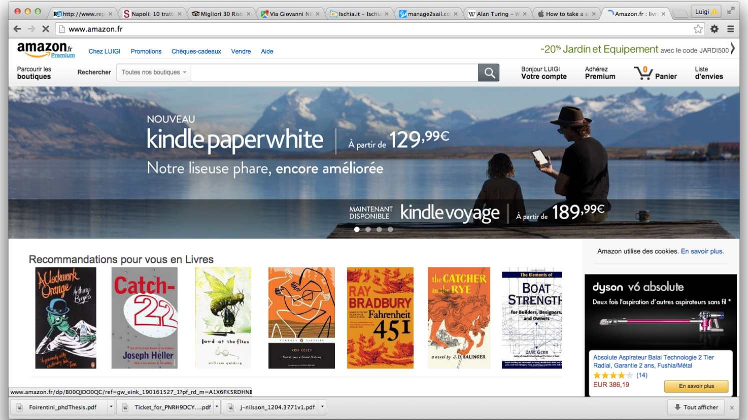
Databases, tables, sqls . . .



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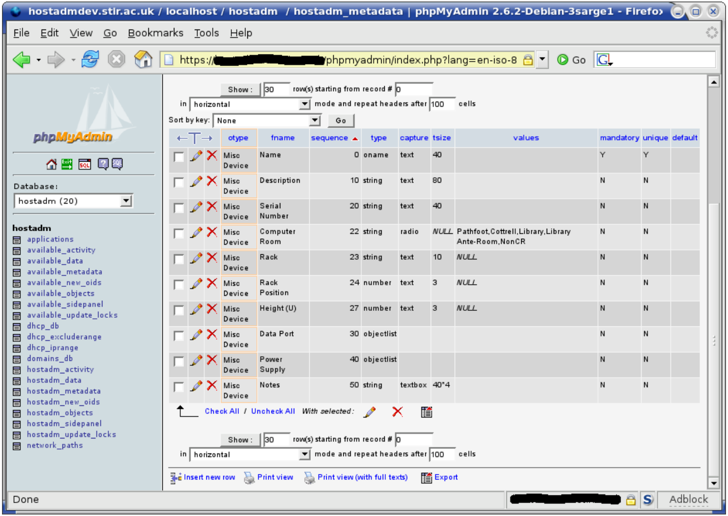

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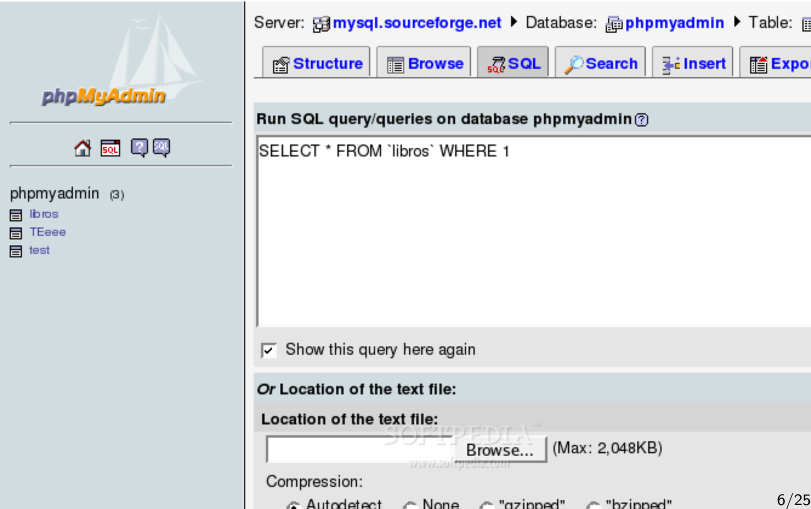

Mixing up tables: the outer join

Name	Surname	Item
Luigi	Santocanale	33
Alan	Turing	21

$\wedge$

Item	Description
33	Book
33	Livre
21	Machine

$=$

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Mixing up tables: the inner union

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$\vee$

Name	Surname	Sport
Diego	Maradona	Football
Usain	Bolt	Athletics

$=$

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Proposition (Spight & Tropashko [3])

The set of tables, whose columns are indexed by a subset of A and values are from a set D, is a lattice, with external join as meet and inner union as join.

Saving the world with lattice theory and logic

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Goals

- Study the equational theory of relational lattices.
- Use knowledge of the equational theory to improve database queries.

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- Study the equational theory of relational lattices.
- Use knowledge of the equational theory to improve database queries.
- Get a job with Oracle,
- . . . a house on the see in California, . . .

Outline

The lattice $\mathrm{R}(D, A)$

A a set of attributes, D a set of values.

A member of $\mathrm{R}(D, A)$ is a pair (X, T) with $X \subseteq A$ and $T \subseteq D^X$.

We have

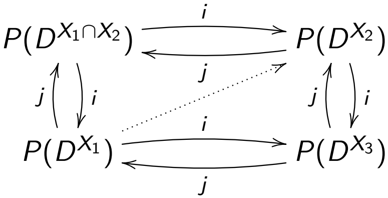
$$(X_1, T_1) \leq (X_2, T_2) \text{ iff } X_2 \subseteq X_1 \text{ and } T_1{\restriction}_{X_2} \subseteq T_2 \,.$$

A bit of categories . . .

R(D, A) is the category of elements of the functor

$$\begin{array}{rcccc} P(A)^{op} & \to & \mathcal{S}\mathrm{et} & \to & \mathcal{SL}_{\vee} \\ X & \mapsto & D^{X} & \mapsto & P(D^{X})\,. \end{array}$$

The image of a pullback square satisfies the Beck-Chevalley property:



A bit of algebra . . .

We have an action

$$j : P(A) \longrightarrow \texttt{Clops}(P(D^A))$$

giving rise to a semidirect product construction:

$$\begin{aligned} \mathsf{R}(D, A) &\equiv P(A) \ltimes_j P(D^A) \\ &:= \{(X, j_X(T)) \mid X \in P(A), T \in D^A\}\,. \end{aligned}$$

This action satisfies the BC-Malcev-property:

$$j_{X \cup Y} = j_X \circ j_Y\,.$$

R(D, A) from a closure operator

Define an ultrametric distance on D^A with values in $P(A)$:

$$\delta(f, g) = \{x \in A \mid f(x) \neq g(x)\} .$$

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This distance is

1. symmetric: $\delta(f, g) = \delta(g, f)$,
2. it has the Beck-Chevalley-Malcev property: *if* $\delta(f, g) \subseteq A \cup B$, *then there exists* h *such that* $\delta(f, h) \subseteq A$ *and* $\delta(h, g) \subseteq B$.

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A subset X of $A + D^A$ is *closed* if $\delta(f, g) \cup \{g\} \subseteq X$ implies $f \in X$.

Proposition (Litak et al. [1])

R(D, A) *is isomorphic to the lattice of closed subsets of* $A + D^A$.

OD-graph based duality

Given a finite latttice L, its OD-graph is the structure $(J(L), \leq, \lhd)$, with

- $(J(L), \leq)$: ordered join-irreducible els.,
- $j \lhd C$ iff $C \subseteq J(L)$ and C is a minimal join-cover of j.

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where

$$X \ll Y \text{ iff } \forall x \in X \exists y \in Y \quad \text{s.t.} \quad x \leq y\,.$$

Use of duality

- semantics, game semantics, . . .
- validity of equations . . .
- counter-model construction . . .
- correspondence results . . .
- heuristics . . .

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- semantics, game semantics, ...
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The lattices $\mathsf{R}(D, A)$ might not be finite, but they are *more-than-perfect*
That is: they enjoy the useful properties of the finite ones.

Minimal join-covers in $\mathsf{R}(D, A)$

$\mathsf{R}(D, A)$ is an atomistic lattice: its atoms are of the form

- $\hat{a}$, for $a \in A$ (these are join-prime);
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Remarkable property:
Each minimal join-cover has at most one non-join-prime element.

Outline

Known equations [1]: AxRL1 and AxRL2

AxRL1 is:

$$x \wedge ((y \wedge (z \vee x)) \vee (z \wedge (y \vee x))) \leq (x \wedge y) \vee (x \wedge z)$$

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For $u \in \{y, z\}$, set

$$\mathtt{d}^{o}_{\ell}(\, u \,) := (u_0 \vee u_1) \wedge (u_0 \vee u_2)\,, \quad \mathtt{d}^{o}_{\rho}(\, u \,) := u_0 \vee (u_1 \wedge u_2)\,.$$

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AxRL2 is:

$$\begin{aligned} x \wedge (\mathtt{d}^{o}_{\ell}(\, y \,) \vee \mathtt{d}^{o}_{\ell}(\, z \,)) & \\ \leq (x \wedge (\mathtt{d}^{o}_{\rho}(\, y \,) \vee \mathtt{d}^{o}_{\ell}(\, z \,))) & \vee (x \wedge (\mathtt{d}^{o}_{\ell}(\, y \,) \vee \mathtt{d}^{o}_{\rho}(\, z \,))) \end{aligned}$$

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Easy proofs that $\mathrm{R}(D, A)$ satifies therse equations using the dual structure.

New equations: UNJP

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Theorem

UNJP holds in a more-than-perfect lattice iff every minimal join-cover contains at most one non-join-prime element.

Remarks

Proposition

AXRL2 is derivable from UNJP, but not the converse.

(Throw `Mace4`, `Prover9`, and `Waldemeister` in the trash ...)

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Proposition

Can derive from UNJP

$$(x \wedge (t_\ell(y) \vee s_\ell(z) \vee w)) \vee (x \wedge (t_\rho(y) \vee s_\rho(z) \vee w)) \\ = (x \wedge (t_\rho(y) \vee s_\ell(z) \vee w)) \vee (x \wedge (t_\ell(y) \vee s_\rho(z) \vee w))$$

whenver $t_\ell(y) = t_\rho(y)$ *and* $s_\ell(z) = s_\rho(z)$ *hold on distributive lattices.*

Other equations: symmetry and the BC property

$$x \wedge (y \vee z) \leq (x \wedge (y \vee (z \wedge (x \vee y)))) \vee (x \wedge (z \vee (y \wedge (x \vee z)))) \quad (\mathrm{SymBC}_1)$$

$$x \wedge ((y \wedge z) \vee (y \wedge x) \vee (z \wedge x)) \leq (x \wedge y) \vee (x \wedge z) \quad (\text{Var-AxRL1})$$

$$x \wedge ((x \wedge y) \vee \mathtt{d}_\ell(z)) \leq (x \wedge ((x \wedge y) \vee \mathtt{d}_\rho(z))) \vee (x \wedge \mathtt{d}_\ell(z)) \quad (\text{R-Mod})$$

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Proposition

UNJP, SymBC$_1$, Var-AxRL1, R-Mod $\vdash$ AxRL1.

Theorem

Assume UNJP. A more-than-perfect lattice satisfies $SymBC_1$, Var-AxRL1, R-Mod if and only if it is symmetric and satisfies the Beck-Chevalley property.

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- A *more-than-perfect* lattice in the variety UNJP is symmetric iff . . .

Theorem

Assume UNJP. A more-than-perfect lattice satisfies $SymBC_1$, Var-AxRL1, R-Mod if and only if it is symmetric and satisfies the Beck-Chevalley property.

- A *more-than-perfect* lattice in the variety UNJP is symmetric iff . . .
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We can obtain similar lattices from (generalized) ultrametric spaces with distance valued on $P(A)$.

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Open problem

Is the above axiomatization complete, w.r.t.

- relational lattices?
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Is the above axiomatization complete, w.r.t.

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- lattices constructed out of ultrametric spaces?

Tentative answer. No, we miss the distance property:

for each join-irreducible elements j, k there exists at most one minimal join-covering $j \lhd C$ such that $k \in C$.

Thanks four your attention . . .

. . . $\mathrm{R}(D, A)$s, get me there !!!