Axiomatising the dual of compact Hausdorff spaces

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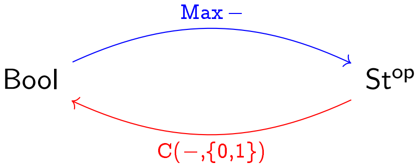
Prologue: Axiomatisability of $\mathsf{KHaus}^{\mathrm{op}}$

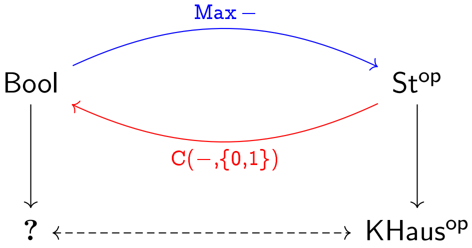
Stone (1936): *"The algebraic theory of Boolean algebras is mathematically equivalent to the topological theory of zero-dimensional compact Hausdorff spaces".*



Bool = Boolean algebras

St = Zero-dimensional compact Hausdorff spaces

Question: Can we *generalise* Stone duality by removing zero-dimensionality?



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KHaus = Compact Hausdorff spaces

Classical solutions:

- Kakutani (1941): M-spaces
- Yosida (1941): (some) abelian ℓ-groups
- Gelfand-Neumark (1943): commutative C*-algebras

Theorem (Duskin, Negrepontis (1969, 1971))

The category $\mathsf{KHaus}^{\mathrm{op}}$ *is monadic over* Set, *i.e.* $\mathsf{KHaus}^{\mathrm{op}}$ *is equivalent to a* λ*-variety.*

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Question: Is there a *finitary* variety of algebras dually equivalent to KHaus?

Answer: No (the monad does not preserve directed colimits).

- Isbell (1982): KHaus is dually equivalent to an $\aleph_1$-variety. An operation of *countably* infinite arity, along with finitely many finitary operations, suffice:

$$\delta(f_1, f_2, f_3, \ldots) := \sum_{i=1}^{\infty} \frac{f_i}{2^i}.$$

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Problem

Give an explicit *tractable* axiomatisation of a variety of infinitary algebras dually equivalent to KHaus (if there is any).

We shall give a *finite* axiomatisation, relying on the theory of MV-algebras.

From C*-algebras to MV-algebras, via ℓ-groups

Theorem (Gelfand-Neumark, 1943)

For every commutative unital C^**-algebra* A *there exists a compact Hausdorff space* X *such that* $A \cong C(X, \mathbb{C})$.

If A is a commutative unital C^*-algebra, define

$$H(A) := \{x \in A \mid x^* = x\}.$$

$H(A)$ is always a unital abelian ℓ-group.

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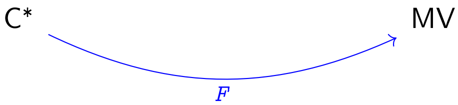
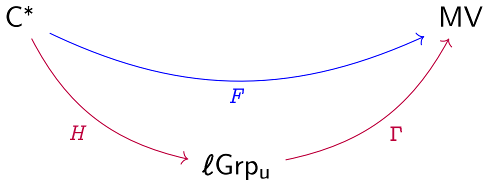


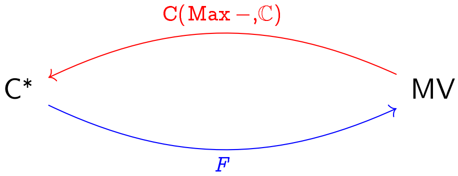
Lemma

The functor F is full and faithful.

$\mathsf{C}^* = \mathrm{C}^*$-algebras
$\mathsf{MV} = \mathrm{MV}$-algebras



$\ell\mathsf{Grp_u}$ = Unital abelian ℓ-groups
C^* = C*-algebras
MV = MV-algebras



Lemma

The functor $\mathrm{C}(\mathrm{Max}\,-,\mathbb{C})$ *is left adjoint to* F*.*

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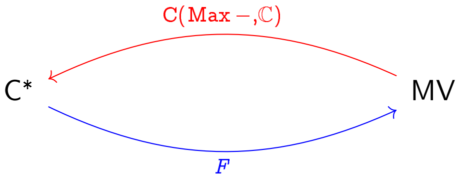

By Gelfand-Neumark duality,

Theorem

The category KHaus^{op} *is equivalent to the reflection of a finitary variety.*

C* = C*-algebras
MV = MV-algebras

δ-algebras

In addition to the MV-algebraic operations $\oplus, \neg, 0$, consider an operation δ of arity $\aleph_0$, whose intended interpretation is the Isbell series

$$\delta(f_1, f_2, f_3, \ldots) = \sum_{i=1}^{\infty} \frac{f_i}{2^i}.$$

For simplicity, define the unary operation

$$f_{\frac{1}{2}}(x) := \delta(x, 0, 0, 0, \ldots).$$

We denote the sequences $x_1, x_2, x_3, \ldots$ and $0, 0, 0, \ldots$ respectively by $\vec{x}, \vec{0}$.

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A **δ-algebra** is a structure $(A, \delta, \oplus, \neg, 0)$ such that $(A, \oplus, \neg, 0)$ is an MV-algebra, and the following identities hold.

△1 $d\left(\delta(\vec{x}), \delta(x_1, \vec{0})\right) = \delta(0, x_2, x_3, \ldots)$.

motivation: $\left(\sum_{i=1}^{\infty} \frac{x_i}{2^i}\right) - \frac{x_1}{2} = \sum_{i=2}^{\infty} \frac{x_i}{2^i}$.

△2 $f_{\frac{1}{2}}(\delta(\vec{x})) = \delta(f_{\frac{1}{2}}(x_1), f_{\frac{1}{2}}(x_2), f_{\frac{1}{2}}(x_3), \ldots)$.

motivation: $\frac{1}{2} \cdot \sum_{i=1}^{\infty} \frac{x_i}{2^i} = \sum_{i=1}^{\infty} \frac{1}{2} \cdot \frac{x_i}{2^i}$.

△3 $\delta(x, x, x, \ldots) = x$.

motivation: $\sum_{i=1}^{\infty} \frac{x}{2^i} = x$.

△4 $\delta(0,\vec{x}) = f_{\frac{1}{2}}(\delta(\vec{x}))$.

motivation: $x_1 = 0 \Rightarrow \sum_{i=1}^{\infty} \frac{x_i}{2^i} = \frac{1}{2} \cdot \sum_{i=2}^{\infty} \frac{x_i}{2^i}$.

△5 $\delta(x_1 \oplus t_1, x_2 \oplus t_2, x_3 \oplus t_3, \ldots) \geqslant \delta(x_1, x_2, x_3, \ldots)$.

motivation: $y_i \geqslant x_i \, \forall i \Rightarrow \sum_{i=1}^{\infty} \frac{y_i}{2^i} \geqslant \sum_{i=1}^{\infty} \frac{x_i}{2^i}$.

△6 $f_{\frac{1}{2}}(x \ominus y) = f_{\frac{1}{2}}(x) \ominus f_{\frac{1}{2}}(y)$.

motivation: $\frac{1}{2}\max(0, x-y) = \max(0, \frac{x}{2} - \frac{y}{2})$.

Let △ be the category whose objects are δ-algebras and whose morphisms are all the MV-homomorphisms preserving the infinitary operation.

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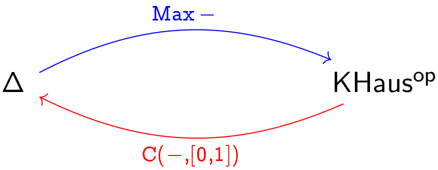
Let Δ be the category whose objects are δ-algebras and whose morphisms are all the MV-homomorphisms preserving the infinitary operation.

Theorem

For every δ-algebra A there exists a compact Hausdorff space X, unique to within a homeomorphism, s.t. $A \cong \mathrm{C}(X, [0,1])$.



Δ = δ-algebras

KHaus = Compact Hausdorff spaces

Theorem

The infinitary variety Δ is equivalent to the category $\mathsf{KHaus}^{\mathsf{op}}$.

Lemma (Step 1)

If A is a δ-algebra, then its MV-algebraic reduct is semisimple.

Idea of the proof.

The radical ideal is closed under the operation δ. If $\mathrm{Rad}\, A$ contains a non-zero infinitesimal element, then there exists a sequence on which δ does not converge. □

Theorem (Chang-Yosida)

An MV-algebra A is semisimple if, and only if, it is isomorphic to a separating subalgebra $\eta(A)$ of $\mathrm{C}(\mathrm{Max}\, A, [0,1])$.

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Lemma (Step 2)

If A is a δ-algebra (identified with $\eta(A)$), then it is dense in $\mathrm{C}(\mathrm{Max}\, A, [0,1])$.

Proof.

We apply a lattice-theoretic version of Stone-Weierstrass theorem. The algebra A is closed under $\wedge, \vee, \oplus$ and contains $1_{\mathrm{Max}\, A}$. If $r \in [0,1]$ and $f \in A$, then rf is shown to be in A. Let $\{r_i\}_{i\in\mathbb{N}} \in \{0,1\}^\omega$ be a dyadic expansion of r. Define $\vec{f} = \{f_i\}_{i\in\mathbb{N}}$ by $f_i := 0_{\mathrm{Max}\, A}$ if $r_i = 0$, and $f_i := f$ if $r_i = 1$. Then $\delta(\vec{f}) = rf$. □

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Axiomatisability of $\mathsf{KHaus}^{\mathsf{op}}$: negative results

Theorem (Banaschewski, 1984)

Let F *be a full subcategory of* KHaus *which extends* St*. If* F *is dually equivalent to an elementary class of finitary algebras closed under products, then* $\mathsf{F} = \mathsf{St}$*.*

Consequence: Stone duality for zero-dimensional compact Hausdorff spaces cannot be extended by retaining the finitary algebraic nature of the dual. In particular, KHaus is not dually equivalent to any finitary variety.

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Recall that an object A of a (locally small) category C is **finitely presentable** if the covariant functor $\hom(A, -)\colon \mathsf{C} \to \mathsf{Set}$ preserves directed colimits.

The category C is **finitely accessible** provided that it has directed colimits and a *dense* subset $\mathcal{A}$ of finitely presentable objects (i.e. every object of C is a directed colimit of objects from $\mathcal{A}$).

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