Distributive contact lattices with nontangential part-of relations

TACL 2015

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Introduction

- Contact relations and nontangential part-of (or proximity, or well-inside) relations
- Boolean case: De Vries, Düntsch, Winter
- Distributive case: Düntsch, MacCaull, Vakarelov, Winter
- Aim: to give an axiomatization linking the two relations in the distributive case, and obtain a Priestley-like duality

Contact algebras: Boolean case

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- C1 $0 \neq a \leq b, c \Rightarrow b \mathcal{C} c$
- C2 $a \mathcal{C} b \Rightarrow b \mathcal{C} a$
- C3 $a \geq b \mathcal{C} c \leq d \Rightarrow a \mathcal{C} d$
- C4 $(a \vee b) \mathcal{C} (c \vee d) \Rightarrow a \mathcal{C} c$ or $a \mathcal{C} d$ or $b \mathcal{C} c$ or $b \mathcal{C} d$

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Well-inside relation: $a \prec b \Leftrightarrow a \perp -b$

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Well-inside relation: $a \prec b \Leftrightarrow a \perp -b$

- DV0 $0 \prec a \prec 1$
- DV1 $a \prec b \Rightarrow a \leq b$
- DV2 $a \prec b \Rightarrow -b \prec -a$
- DV3 $a \leq b \prec c \leq d \Rightarrow a \prec d$
- DV4 $a, c \prec b, d \Rightarrow a \vee c \prec b \wedge d$

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Well-inside relation: $a \prec b \Leftrightarrow a \perp -b$
Dual-contact: $a\,\breve{\mathcal{C}}\,b \Leftrightarrow -a\,\mathcal{C} - b$
Dual-well-inside: $a \breve{\prec} b \Leftrightarrow -a \perp b$

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$$\begin{array}{rcll} f(a) & = & \{b \in B \mid a \prec b\} & \qquad B \to B \\ g(a) & = & \{b \in B \mid a \perp b\} & \qquad B \to B^{\partial} \end{array}$$

Distributive case

Priestley duality

D distributive lattice $\longrightarrow (X_D, \tau, \leq)$

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Dual equivalence between

- **Dist**: bounded distributive lattices + homomorphisms
- **Pries**: Priestley spaces + continuous order-preserving maps

Distributive case

Duality for multi-operators

We consider a distributive lattice D endowed with two relations $\mathcal{C}$ and $\prec$ satisfying

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- DV0 $0 \prec a \prec 1$
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Distributive case

Duality for multi-operators

Let's consider a distributive lattice D endowed with $\mathcal{C}$ and $\prec$ and satisfying C0, C3, C4, DV0, DV3, DV4.

$f(a) = \{b \in D \mid a \prec b\}$

$g(a) = \{b \in D \mid a \perp b\}$

Distributive case

Duality for multi-operators

Let's consider a distributive lattice D endowed with $\mathcal{C}$ and $\prec$ and satisfying C0, C3, C4, DV0, DV3, DV4.
$f(a) = \{b \in D \mid a \prec b\}$
$g(a) = \{b \in D \mid a \perp b\}$
These multi-operators yield on the dual two relations R and S:

$$x \, R \, y \Leftrightarrow \bigcup f(F_x) \subset F_y$$

$$x \, S \, y \Leftrightarrow \bigcup g(F_x) \subset I_y$$

Distributive case

Duality for multi-operators

The relations R and S are closed and satisfy

- $\geq \circ R \circ \geq = R$
- $\geq \circ S \circ \leq = S$

Distributive case

Duality

Theorem (Hansoul)

$(D, \mathcal{C}, \prec)$ can be embedded in a Boolean contact algebra iff $R = (R \cap S) \circ \geq$ and $S = (R \cap S) \circ \leq$.

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When can we recover R and S knowing $T = R \cap S$?

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When can we recover R and S knowing $T = R \cap S$?
The corresponding problem for $\Box$ and $\Diamond$ has been examined by Přenosil

Distributive case

Duality

Theorem

$R = T \circ \geq$ and $S = T \circ \leq$ iff $(D, \mathcal{C}, \prec)$ satisfies

CDV1 $a \prec b \vee c, a \perp c \Rightarrow a \prec b$

CDV2 $a \perp b \wedge c, a \prec b \Rightarrow a \perp c$

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Definition

$(D, \mathcal{C}, \prec)$ *mereotopological lattice* if it satisfies $CDV1$ and $CDV2$ (in addition to the original axioms).

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Fact

If $(D, \mathcal{C}, \prec)$ is a mereotopological lattice, then $T = R \cap S$ is closed, left-increasing ($T = \geq \circ T$) and right-convex ($T = (T \circ \geq) \cap (T \circ \leq)$).

Distributive case

Duality

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$R = T \circ \geq$, $S = T \circ \leq$.

$$\prec_T = \{(U, V) \in D_X^2 \mid R[U, -] \subset V\}$$

$$\mathcal{C}_T = \{(U, V) \in D_X^2 \mid S[U, -] \not\subset V^c\}$$

Distributive case

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X Priestley space, T closed left-increasing right-convex relation on X.
$R = T \circ \geq$, $S = T \circ \leq$.

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$$\mathcal{C}_T = \{(U, V) \in D_X^2 \mid S[U, -] \not\subset V^c\}$$

Theorem

$(D_X, \mathcal{C}_T, \prec_T)$ is a mereotopological lattice.

Distributive case

Morphisms

Mereo. lattices $\longleftrightarrow$ Priestley spaces $+$ clirc relations

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Morphisms?

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Morphisms?

Lattice homomorphisms $h : D \to D'$ s.t.

- $a \prec b \Rightarrow h(a) \prec' h(b)$
- $a \perp b \Rightarrow h(a) \perp' h(b)$

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Morphisms

Mereo. lattices $\longleftrightarrow$ Priestley spaces + clirc relations
Morphisms?

Lattice homomorphisms $h : D \to D'$ s.t.

- $a \prec b \Rightarrow h(a) \prec' h(b)$
- $a \perp b \Rightarrow h(a) \perp' h(b)$

Continuous order-preserving maps $\varphi : X \to X'$ s.t.

- $x \, T \, y \Rightarrow \varphi(x) \, T' \, \varphi(y)$

Distributive case

p-morphisms

$h : (D, f) \to (D', f')$ homomorphism s.t.

- $h(f(a)) = f'(h(a))$

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p-morphisms

$h : (B, f) \to (B', f')$ homomorphism s.t.

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$\varphi : (X, R) \to (X', R')$ continuous map s.t.

- $\varphi(R[-, y]) = R'[-, \varphi(y)]$

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$h : (D, f) \to (D', f')$ homomorphism s.t.

- $h(f(a))\uparrow = f'(h(a))$

$\varphi : (X, R) \to (X', R')$ continuous, order-pres. map s.t.

- $\varphi(R[-, y])\uparrow = R'[-, \varphi(y)]$

Distributive case

p-morphisms

In our case:

$h : (D, \mathcal{C}, \prec) \to (D', \mathcal{C}', \prec')$ homomorphism s.t.

- $h(a) \prec' b' \Leftrightarrow \exists b \; a \prec b$ and $h(b) \leq b'$
- $h(a) \perp' b' \Leftrightarrow \exists b \; a \perp b$ and $h(b) \leq b'$

$\varphi : (X, T) \to (X', T')$ continuous, order-pres. map s.t.

- $\varphi(R[-, y])\uparrow = R'[-, \varphi(y)]$
- $\varphi(S[-, y])\uparrow = S'[-, \varphi(y)]$

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- De Vries-like duality

Thank you![1]

[1] and thanks to R. Cavus, L. De Rudder and G. Hansoul

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