Bilattice Logic of Epistemic Action and Knowledge

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Introduction: DEL

- Dynamic logics are language expansions of modal logic designed to reason about change, and widely applied in computer science.
- Dynamic epistemic logic (DEL) models changes affecting the cognitive state of agents.
- We focus here on changes that do not concern facts of the world but rather cognitive states (e.g., public announcements).
- Logical consequence in DEL can be difficult to treat (from a syntactic as well as semantic point of view), e.g. because it is not substitution-invariant.

Introduction: DEL

- Recent work of Alessandra Palmigiano and collaborators tackles the above problems using display calculi to axiomatize systems of DEL and duality theory to study their semantics.
- Alessandra & co. propose a uniform methodology for developing DEL in a number of non-classical settings, which can be useful for different applications.
- In this talk I will report on the algebraic and duality-theoretic aspects of this ongoing enterprise.

Epistemic updates

- Epistemic change is represented in DEL as a transformation from a (relational, algebraic) model representing the current situation to a new model that represents the situation after some epistemic action has occurred.
- The update on the epistemic state of agents caused by an action is known as epistemic update.
- Epistemic updates are formalized
  - on Kripke-style models via (pseudo-) co-products and sub-models,
  - on algebras via (pseudo-) products and quotients.

Epistemic Action and Knowledge

- The logic EAK was introduced by A. Baltag, L.S. Moss and S. Solecki (1999) to deal with "Public Announcements, Common Knowledge and Private Suspicions".
- The language of EAK is that of modal logic (S5) expanded with dynamic operators $\langle \alpha \rangle$ and $[\alpha]$, where α is an action structure.
- The intended meaning of $\langle \alpha \rangle \varphi$ is: the action α can be executed, and after execution φ holds.
- Dually, $[\alpha]\varphi$ means: if the action α can be executed, then after execution φ holds.

Epistemic Action and Knowledge

Language of (classical, single-agent) EAK

$$\varphi ::= p \in \mathsf{Var} \mid \neg\varphi \mid \varphi \vee \varphi \mid \ldots \mid \Diamond\varphi \mid \Box\varphi \mid \langle\alpha\rangle\varphi \mid [\alpha]\varphi,$$

where α is an action structure:

$$\alpha = (K, k, R_\alpha, Pre_\alpha : K \to Fm).$$

Kripke semantics

For $M = (W, R, v)$, define

$$M, w \Vdash \langle\alpha\rangle\varphi \quad \text{iff} \quad M, w \Vdash Pre(\alpha) \text{ and } M^\alpha, w \Vdash \varphi$$

$$M, w \Vdash [\alpha]\varphi \quad \text{iff} \quad \text{if } M, w \Vdash Pre(\alpha), \text{ then } M^\alpha, w \Vdash \varphi$$

where M^α is the updated model, after execution of α.

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Epistemic Action and Knowledge

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Epistemic update

Intermediate model (pseudo coproduct)

Given $\alpha := (K, k, R_\alpha, Pre_\alpha : K \to Fm)$ and $M = (W, R, v)$, let

$$\coprod_\alpha M := (\coprod_K W, R \times R_\alpha, \coprod_K v)$$

- $\coprod_K W \cong W \times K$
- $(w, j)(R \times \alpha)(u, i)$ iff wRu and $jR_\alpha i$
- $(\coprod_K v)(p) := \coprod_K v(p)$.

The second step, M^α

M^α is the submodel of $\coprod_\alpha M$ with domain

$$W^\alpha := \{(w, j) \mid M, w \Vdash Pre_\alpha(j)\}.$$

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Epistemic update

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Axiomatization

EAK is axiomatized by

- the axioms and rules of modal logic (S5) plus the following axioms:

  1. $\langle\alpha\rangle p \leftrightarrow (Pre(\alpha) \wedge p)$ where $p \in \mathsf{Var}$
  2. $\langle\alpha\rangle\neg\varphi \leftrightarrow (Pre(\alpha) \wedge \neg\langle\alpha\rangle\varphi)$
  3. $\langle\alpha\rangle(\varphi \vee \psi) \leftrightarrow (\langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi)$
  4. $\langle\alpha\rangle\Diamond\varphi \leftrightarrow (Pre(\alpha) \wedge \bigvee\{\Diamond\langle\alpha_i\rangle\varphi \mid kR_\alpha i\})$

  where $\alpha = \alpha_k$ and $\alpha_i = (K, i, R_\alpha, Pre_\alpha)$ for each $i \in K$.

- The rule:

$$\text{from } \emptyset \vdash \varphi \rightarrow \psi \quad \text{infer} \quad \emptyset \vdash \langle\alpha\rangle\varphi \rightarrow \langle\alpha\rangle\psi.$$

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Methodology: dual characterizations

Non-classical
Algebraic Semantics

$\mathbb{A} \twoheadleftarrow \prod_{\alpha} \mathbb{A} \twoheadrightarrow \mathbb{A}^{\alpha}$

Non-classical
Relational Semantics

$M \hookrightarrow \coprod_{\alpha} M \hookleftarrow M^{\alpha}$

Classical
Algebraic Semantics

$\mathbb{A} \twoheadleftarrow \prod_{\alpha} \mathbb{A} \twoheadrightarrow \mathbb{A}^{\alpha}$

Classical
Relational Semantics

$M \hookrightarrow \coprod_{\alpha} M \hookleftarrow M^{\alpha}$

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Intermediate models as algebras

Let $\mathbb{A}$ be a modal algebra and $\alpha = (K, k, R_\alpha, Pre_\alpha : K \to \mathbb{A})$ an action structure over $\mathbb{A}$.

Define

$$\prod_\alpha \mathbb{A} := (\mathbb{A}^K, \Diamond^{\Pi_\alpha \mathbb{A}}, \Box^{\Pi_\alpha \mathbb{A}})$$

where, for each $f : K \to \mathbb{A}$ and $j \in K$,

$$(\Diamond^{\Pi_\alpha \mathbb{A}} f)(j) = \bigvee \{\Diamond^{\mathbb{A}} f(i) \mid jR_\alpha i\}$$

$$(\Box^{\Pi_\alpha \mathbb{A}} f)(j) = \bigwedge \{\Box^{\mathbb{A}} f(i) \mid jR_\alpha i\}.$$

A similar definition can be given for (semi)lattices with operators, HAOs, modal bilattices etc.

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The pseudo quotient

Let $\mathbb{A}$ be a modal algebra and $\alpha = (K, k, R_\alpha, Pre_\alpha : K \to \mathbb{A})$ an action structure over $\mathbb{A}$.

Since $Pre_\alpha \in \prod_\alpha \mathbb{A}$, we let, for every $b, c \in \prod_\alpha \mathbb{A}$,

$$b \equiv_\alpha c \quad \text{iff} \quad b \wedge Pre_\alpha = c \wedge Pre_\alpha$$

and we have a Boolean algebra $\prod_\alpha \mathbb{A}/\equiv_\alpha$ on which we define, for any $[b] \in \prod_\alpha \mathbb{A}/\equiv_\alpha$,

$$\Diamond^\alpha [b] := [\Diamond^{\prod_\alpha \mathbb{A}}(Pre_\alpha \wedge b)]$$

$$\Box^\alpha [b] := [\Box^{\prod_\alpha \mathbb{A}}(Pre_\alpha \to b)].$$

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The pseudo quotient

Remarks

- We can define an injective map $\iota\colon [b] \longmapsto b \wedge Pre_\alpha$ that embeds $\prod_\alpha \mathbb{A}/{\equiv_\alpha}$ into $\prod_\alpha \mathbb{A}$.
- If $\mathbb{A}$ is a different algebra with operators (bilattice, MV), the relation $\{(b, c) \in A \times A \mid b \wedge Pre_\alpha = c \wedge Pre_\alpha\}$ may not be a congruence of the non-modal reduct of $\mathbb{A}$.
- A more widely applicable recipe: if the underlying non-modal logic $\mathcal{L}$ is algebraizable, take the congruence $\theta(Fi_\mathcal{L}(Pre_\alpha))$ determined by the logical filter $Fi_\mathcal{L}(Pre_\alpha)$.
- To define $\Diamond^\alpha, \Box^\alpha$ we still need a uniform characterization of $\theta(Fi_\mathcal{L}(Pre_\alpha))$, for example

$$\{(b, c) \in A \times A \mid b \wedge (Pre_\alpha)^n = c \wedge (Pre_\alpha)^n\}$$

works for n-potent modal MV-algebras.

Algebraic semantics

For every algebraic model $M = (\mathbb{A}, v)$, where $v \colon \mathrm{Var} \to \mathbb{A}$, the extension map $[\![\cdot]\!]_M : Fm \to \mathbb{A}$ is defined as:

$$
\begin{array}{rcll}
[\![p]\!]_M & = & v(p) & \\
[\![\varphi \spadesuit \psi]\!]_M & = & [\![\varphi]\!]_M \spadesuit^{\mathbb{A}} [\![\psi]\!]_M & \text{for } \spadesuit \in \{\wedge, \vee, \to, \ldots\} \\
[\![\heartsuit \varphi]\!]_M & = & \heartsuit^{\mathbb{A}} [\![\varphi]\!]_M & \text{for } \heartsuit \in \{\Diamond, \Box, \neg, \ldots\} \\
[\![\langle\alpha\rangle\varphi]\!]_M & = & [\![Pre(\alpha_k)]\!]_M \wedge^{\mathbb{A}} \pi_k \circ \iota([\![\varphi]\!]_{M^\alpha}) & \\
[\![[\alpha]\varphi]\!]_M & = & [\![Pre(\alpha_k)]\!]_M \to^{\mathbb{A}} \pi_k \circ \iota([\![\varphi]\!]_{M^\alpha}). &
\end{array}
$$

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Completeness results

- Soundness of the axioms is checked w.r.t. to algebraic models.
- Completeness is obtained using the interaction axioms to reduce EAK to its static fragment (e.g., modal logic S5).
- Soundness and completeness w.r.t. relational models follow by duality.
- Classical and intuitionistic EAK are also axiomatized by means of modular, cut-free display-style sequent calculi.

Further work

- Understand epistemic updates on algebras in the most general setting (role of Leibniz congruence, preservation of equations).
- Extend to other logics (e.g., positive modal logic, infinite-valued Łukasiewicz, logics of order).
- Study updates in a topological duality setting.
- Applications in non-classical reasoning (e.g., public announcements with lies).