

MV-pairs and state operators

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Introduction

States on MV-algebras as a generalization of probability measures on Boolean algebras — *Mundici, 1995.*

Flaminio, Montagna, 2009—*state operator on MV-algebras* — unary operation with basic properties of states — *state MV-algebras*.

Effect algebras Foulis, Bennett, 1994—modelling unsharp measurements in QM.

Subclasses of EAs: orthomodular posets and lattices, orthoalgebras, *MV-algebras*.

Jenča, 2007 — relations between MV-algebras and Boolean algebras via *MV-pairs*.

We study *relations between state operators and MV-pairs*.

Outline

1. We introduce some definitions and known results that we need in what follows.
2. We introduce a simpler *definition of morphisms of MV-pairs* equivalent with the original definition introduced by *di Nola, Holčapek, Jenča, 2009*.
3. We introduce the *notions of state MV-pairs and strong state MV-pairs*, and study their relations with state MV-algebras.
4. We find conditions under which an MV-pair, resp. a state MV-pair, gives rise to a *subdirectly irreducible MV-algebra*, resp. a *subdirectly irreducible state-MV-algebra*.

Effect algebras

An *effect algebra* (EA) is a partial algebra $(E; \oplus, 0, 1)$ with a partially defined binary operation \oplus and two constants 0 and 1 such that

- (E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined, and
$$a \oplus b = b \oplus a.$$
- (E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined, and
$$a \oplus (b \oplus c) = (a \oplus b) \oplus c.$$
- (E3) $\forall a \in E : \exists! a' \in E$ such that $a \oplus a' = 1.$
- (E4) If $a \oplus 1$ is defined, then $a = 0.$
 - partial order: $a \leq b$ iff $\exists c \in E : a \oplus c = b,$
 $0 \leq a \leq 1 \quad \forall a \in E, \exists a \oplus b \Leftrightarrow a \leq b' \Leftrightarrow a \perp b.$

MV-effect algebras

An *MV-algebra* is an algebra $(M; \boxplus, ', 0)$ of type $(2, 1, 0)$, such that \boxplus is commutative, associative with neutral element 0 , $x'' = x$, $x \oplus 0' = 0'$, and $(x' \boxplus y)' \boxplus y = (y' \boxplus x)' \boxplus x$.

MV-algebra is a distributive lattice with 0 the smallest and $1 := 0'$ the greatest element.

MV-effect algebra is a lattice ordered effect algebra satisfying the *Riesz decomposition property (RDP)*: $\forall a, b, c \in E$, if $a \leq b \oplus c$, then there are $b_1, c_1 \in E$ such that $b_1 \leq b$, $c_1 \leq c$ and $a = b_1 \oplus c_1$.

MV-effect algebras are in 1-1 correspondence with MV-algebras.

Congruences on EAs

A relation \sim on an effect algebra E is a *congruence* iff:

(C1) \sim is an equivalence relation.

(C2) $a_1 \sim a_2, b_1 \sim b_2, a_1 \perp b_1, a_2 \perp b_2 \Rightarrow a_1 \oplus b_1 \sim a_2 \oplus b_2$.

(C3) $a \sim c, a \perp b \Rightarrow \exists d \in E : d \sim b, d \perp c$:

$$\begin{array}{ccc} a \oplus b \\ \sim & \sim \\ c \oplus d \end{array}$$

- $E/\sim := \{[a] : a \in E\}$ is an EA:

$$\begin{aligned} [a] \perp [b] &\Leftrightarrow \exists a_1 \in [a], b_1 \in [b], a_1 \perp b_1, \\ [a] \oplus [b] &= [a_1 \oplus b_1]. \end{aligned}$$

- A congruence preserves RDP.

MV-pair

B – Boolean algebra, G – subgroup of $\text{Aut}(B)$,
 (B, G) — *BG-pair*.

$$L^+(a, b) := \{g(a) \wedge f(b) : f, g \in G\}$$

$\max L^+(a, b)$ — maximal elements in $L^+(a, b)$,

A *BG-pair* is an **MV-pair**^a iff

(MVP1) $\forall a, b \in B, f \in G, a \perp b, a \perp f(b) \exists h \in G, h(a \vee b) = a \vee f(b)$.

(MVP2) $\forall a, b \in B, x \in L^+(a, b), \exists m \in \max L^+(a, b), x \leq m$.

^aJenča 2007

Boolean – MV

(B, G) – MV-pair, $a, b \in B$:

$$a \sim_G b \Leftrightarrow \exists f \in G : b = f(a).$$

Let (B, G) be an MV-pair. Then

- (MVP1) implies B/\sim_G is a effect algebra;
- (MVP2) implies that B/\sim_G is a lattice and

$$[a] \wedge [b] = \max L^+(a, b)$$

("=" is the set equality)

MV – Boolean

- M – MV-algebra – bounded distributive lattice
 $\implies \exists$ a unique Boolean algebra $B(M)$ – *R-generated Boolean algebra* – such that M is a 0, 1-sublattice and M generates $B(M)$ as a boolean ring.

$\forall x \in B(M), \exists x_1 \leq x_2 \leq \dots \leq x_n \in M:$

$x = x_1 + x_2 + \dots + x_n$ (+ is symmetric difference) – *chain representation* of x .

- It can be arranged n - even, and the map

$$\phi_M : B(M) \rightarrow M$$

$$\phi_M(x) = \bigoplus_{i=1}^n (x_{2i} \ominus x_{2i-1}),$$

where $\{x_i\}_{i=1}^{2n}$ is an M -chain representation of x , is a surjective morphism of effect algebras^a.

^aJenca 2004

R-generated — MV-pair

M - MV-algebra;

$$G(M) := \{f \in \text{Aut}(B(M)) : \\ \forall x \in B(M), \phi_M(x) = \phi_M(f(x))\}.$$

• (a) $(B(M), G(M))$ is an MV-pair.

(b) $\forall x, y \in B(M), x \sim_{G(M)} y$ iff $\phi_M(x) = \phi_M(y)$.

(c) $\beta_M : B(M)/G(M) \rightarrow M,$

$$\beta_M([x]_{G(M)}) = \phi_M(x),$$

is an isomorphism of MV-algebras^a.

^aJenča 2007

Morphisms of MV-pairs

$(B_1, G_1), (B_2, G_2)$ — MV-pairs, $\psi_B : B_1 \rightarrow B_2$ is a *morphism of MV-pairs*^a iff:

(1) ψ_B is a morphism of Boolean algebras,

(2) $\forall x \in B_1, \forall f \in G_1, \exists g \in G_2:$

$$\psi_B(f(x)) = g(\psi_B(x)),$$

(3) $\forall a, b \in B_1, m \in \max L^+(a, b) \implies$

$$\psi_B(m) \in \max L^+(\psi_B(a), \psi_B(b)).$$

• (B, G) — MV-pair, $x \in B$:

$O(x) := \{y \in B : \exists f \in G, y = f(x)\}$ — *orbit of x* .

• A Boolean algebra morphism ψ_B is a morphism of MV-pairs iff it maps orbits to orbits and maximal elements to maximal elements in L^+ .

^aDi Nola, Holčapek, Jenča 2009

MV-pairs – MV-algebras

- $\psi : M_1 \rightarrow M_2$ — morphism of MV-algebras $\implies \nabla(\psi) : B(M_1) \rightarrow B(M_2)$ — (unique) morphism of Boolean algebras.

For every morphism ψ of MV-algebras,

$\nabla(\psi) : B(M_1) \rightarrow B(M_2)$ is morphism of MV-pairs.

- $\psi_B : (B_1, G_1) \rightarrow (B_2, G_2)$ — morphism of MV-pairs $\implies \Delta(\psi_B) : B_1/G_1 \rightarrow B_2/G_2$,
 $\Delta(\psi_B)([x]_{G_1}) := [\psi_B(x)]_{G_2}$

For every morphism ψ_B of MV-pairs, $\Delta(\psi_B)$ is morphism of MV-algebras.

State MV-algebras

A *state MV-algebra*^a is a pair (M, σ) ,
 M — MV-algebra, $\sigma : M \rightarrow M$:

(1) $\sigma(1) = 1$.

(2) $\sigma(x') = \sigma(x)'$.

(3) $\sigma(x \boxplus y) = \sigma(x) \boxplus \sigma(y \boxminus (x \boxdot y))$,

where $x \boxdot y = (x' \boxplus y')'$.

(4) $\sigma(\sigma(x) \boxplus \sigma(y)) = \sigma(x) \boxplus \sigma(y)$.

- The class of state MV-algebras forms a variety.
- $\sigma(\sigma(x)) = \sigma(x)$.
- $\sigma(M)$ is MV-subalgebra of M .

^aFlaminio, Montagna 2009

State operators on EAs

$(E, \oplus, 0, 1)$ – effect algebra, $\sigma : E \rightarrow E$ is a *state operator*^a iff

$$(\sigma 1) \quad \sigma(1) = 1.$$

$$(\sigma 2) \quad \sigma(a \oplus b) = \sigma(a) \oplus \sigma(b) \text{ if } \exists(a \oplus b).$$

$$(\sigma 3) \quad \sigma(\sigma(a)) = \sigma(a).$$

A state operator is *strong* iff

$$(\sigma 4) \quad \sigma(\sigma(a) \wedge \sigma(b)) = \sigma(a) \wedge \sigma(b) \text{ if } \exists(\sigma(a) \wedge \sigma(b)).$$

$(E, \oplus, 0, 1, \sigma)$ – (*strong*) state effect algebra.

- For MV-effect algebra, this definition of σ coincides with the definition of *Flaminio, Montagna 2009* iff σ is strong.

^aBuhagiar, Chetcuti, Dvurečenskij 2011

MV-pair and state operator

- (B, G) – MV-pair, $\sigma_B : B \rightarrow B$ – state operator.

Then:

$$\sigma_{B*}([a]_G) := [\sigma_B(a)]_G$$

is a state operator on $M = B/G$ iff

$$(O) \quad \sigma_B(O(a)) \subseteq O(\sigma_B(a)), \quad a \in B.$$

If σ_B is strong and equality in (O) holds, the σ_{B*} is strong.

- M – MV-effect algebra,

$\sigma : M \rightarrow M$ — (strong) state operator on M . Then:

$$\sigma^*(a) := \sigma(\phi_M(a)), \quad a \in B(M)$$

is a (strong) state operator on $B(M)$.

state MV-pair

(B, G, σ) is a *state MV-pair* iff

(B, G) is an MV-pair, $\sigma : B \rightarrow B$ is a state operator, and condition (O) holds.

A state MV-pair (B, G, σ) is a *strong* iff σ is strong state operator and equality in (O) holds.

- (B, G, σ) — (strong) state-MV-pair $\implies M = B/G$ — (strong) state MV-effect algebra with $\sigma_*([a]_G) = [\sigma(a)]_G$.
- (M, σ) — (strong) state MV-algebra $\implies (B(M), G(M), \sigma^*)$ — (strong) state-MV-pair with $\sigma^*(a) = \sigma(\phi_M(a))$, and $(\sigma^*)_* = \sigma$.

Subdirect irreducibility 1

An MV-algebra M is subdirectly irreducible iff M has a smallest nontrivial ($\neq \{0\}$) ideal I_0 .

(B, G) — MV-pair, an ideal J in B is *G-invariant* iff

$$a \in J \implies f(a) \in J, f \in G.$$

- Let (B, G) be an MV-pair and $M := B/G$ the corresponding MV-algebra. Then M is subdirectly irreducible iff B has a smallest nontrivial G -invariant ideal.

M — MV-algebra, I — ideal in M , then $I^* := \phi_M^{-1}(I)$ — $G(M)$ -invariant ideal in $B(M)$.

- Let M be an MV-algebra, $(B(M), G(M))$ the corresponding MV-pair. Then M is subdirectly irreducible with a smallest nontrivial ideal I_0 iff $B(M)$ has a smallest nontrivial $G(M)$ -invariant ideal I_0^* that extends I_0 .

Subdirect irreducibility 2

(M, σ) — state-MV-algebra, an ideal $I \subseteq M$ is a *σ -ideal* iff

$$a \in I \implies \sigma(a) \in I.$$

- (M, σ) — *state-MV-effect algebra*,
 $(B(M), G(M), \sigma^*)$ — *the corresponding state-MV-pair*. Then (M, σ) is subdirectly irreducible iff $B(M)$ has a smallest nontrivial $G(M)$ -invariant σ^* -ideal.
- (B, G, σ) — *state-MV-pair*, $(B/G, \sigma_*)$ — *the corresponding state-MV-effect algebra*. Then $(B/G, \sigma_*)$ is subdirectly irreducible iff B has a smallest nontrivial G -invariant σ -ideal.

Morphism of state-MV-pairs

$(M_1, \sigma_1), (M_2, \sigma_2)$ — state MV-algebras,

$\psi : M_1 \rightarrow M_2$ is *morphism of state MV-algebras*

iff ψ is a morphism of MV-algebras such that

$$\psi \circ \sigma_1 = \sigma_2 \circ \psi.$$

$(B_1, G_1, \sigma_1), (B_2, G_2, \sigma_2)$ — state-MV-pairs,

$\psi_B : B_1 \rightarrow B_2$ is a *morphism of state-MV-pairs*

iff ψ_B is a morphism of MV-pairs such that:

$$\psi_B \circ \sigma_1 = \sigma_2 \circ \psi_B.$$

- ψ — state MV-algebra morphism of state-MV-effect algebras (M_1, σ_1) and (M_2, σ_2) . Then $\nabla(\psi)$ is a morphism of state-MV-pairs $(B(M_1), G(M_1), \sigma_1^*)$ and $(B(M_2), G(M_2), \sigma_2^*)$.
- ψ_B — morphism of state MV-pairs (B_1, G_1, σ_1) and (B_2, G_2, σ_2) . Then $\Delta(\psi_B)([a]_{G_1}) = [\psi_B(a)]_{G_2}$ is a morphism state MV-algebras $(B_1/G_1, \sigma_{1*})$ and $(B_2/G_2, \sigma_{2*})$.

diagram

$$\begin{array}{ccc} \sigma_1(M_1) & \xrightarrow{\psi \equiv \nabla(\psi)} & \sigma_2(M_2) \\ \sigma_1 \uparrow & & \uparrow \sigma_2 \\ M_1 & \xrightarrow{\psi} & M_2 \\ \phi_{M_1} \uparrow & & \uparrow \phi_{M_2} \\ B(M_1) & \xrightarrow{\nabla(\psi)} & B(M_2) \end{array}$$

$$\psi(\sigma_1(\phi_{M_1}(a))) = \sigma_2(\phi_{M_2}(\nabla(\psi)(a))) = \sigma_2(\psi(\phi_{M_1}(a)))$$

Functors

$$\nabla(M, \sigma) = (B(M), G(M), \sigma^*),$$
$$\Delta(B, G, \sigma) = (B/G, \sigma_*)$$

- ∇ and Δ are functors;

$$\Delta \nabla(M, \sigma) \simeq (M, \sigma).$$

References

- [1] S. Pulmannová, E. Vinceková: *MV-pairs and state operators*, Fuzzy Sets and Systems **260** (2015), 62–76.
- [2] D. Buhagiar, E. Chetcuti, A. Dvurečenskij: *Loomis-Sikorski theorem and Stone duality for effect algebras with internal state*, Fuzzy Sets and Systems **172** (2011), 71–86.
- [3] C.C. Chang: *Algebraic analysis of many-valued logics*, Trans. amer. Math. Soc. **88** (1958), 467–490.
- [4] A. Di Nola, M. Holčapek, G. Jenča: *The category of MV-pairs*, Logic Journal of the IGLP **17** (2009), 395–412.
- [5] A. Dvurečenskij, S. Pulmannová: *New Trends in Quantum Structures*. Kluwer Academic Publishers, Dordrecht, 2000.
- [6] T. Flaminio, F. Montagna: *MV-algebras with internal states and probabilistic fuzzy logics*, International Journal of Approximate Reasoning **50** (2009), 138–152.
- [7] D.J. Foulis, M.K. Bennett: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325–1346.
- [8] G. Grätzer: *General Lattice Theory*. Birkhäuser, second edition, 1998.
- [9] G. Jenča: *A representation theorem for MV-algebras*, Soft Computing **11** (2007), 557–564.
- [10] G. Jenča: *Boolean algebras R-generated by MV-effect algebras*, Fuzzy sets and systems **145** (2004), 279–285.
- [11] D. Mundici: *Averaging the truth-value in Łukasiewicz logic*, Studia Logica **55** (1995), 113–127.
- [12] T. Vetterlein: *Boolean algebras with an automorphism group: a framework for Łukasiewicz logic*, J. Mult.-Val. log. Soft comput. **14** (2008), 51–67.