Tense operators in logics without negation

Jan Paseka
Department of Mathematics and Statistics
Masaryk University
Brno, Czech Republic
paseka@math.muni.cz

Ivan Chajda
Department of Algebra and Geometry
Palacký University Olomouc
Olomouc, Czech Republic
ivan.chajda@upol.cz

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Outline

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Introduction - tense operators on distributive lattices

For Boolean algebras, the so-called tense operators were already introduced by Burges. Tense operators express the quantifiers “it is always going to be the case that” and “it has always been the case that” and hence enable us to express the dimension of time in the logic. In this lecture we introduce tense operators on distributive lattices.

A crucial problem concerning tense operators is their representation. Having a Boolean algebra with tense operators, it is well known that there exists a time frame such that each of these operators can be obtained by their construction for two-element Boolean algebra $\{0, 1\}$. We solved this problem with I. Chajda for such tense operators on distributive lattices that form the so-called MN-positive pairs.

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Tense operators on $[0,1]$

Tense operators were used to express the dimension of time in logics.

- Let T be a time scale,
- then elements $f(t)$ from $[0,1]^T$ correspond to the evaluation of the validity of the formula f in time.

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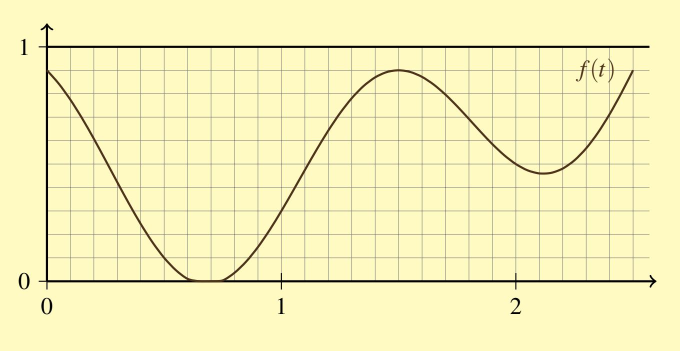

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On the time scale T we will introduce a relation $R \subseteq T^2$.

- xRy means that the moment x is before the moment y.

Moreover, we introduce operators G a H on $[0,1]^T$ as follows:

- Gf means that f will be true in future with at least the same degree as f is now.
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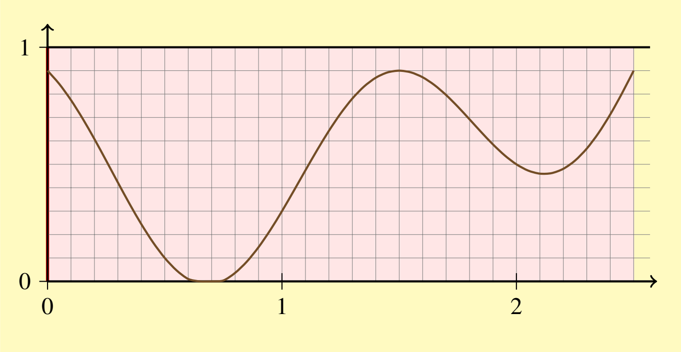

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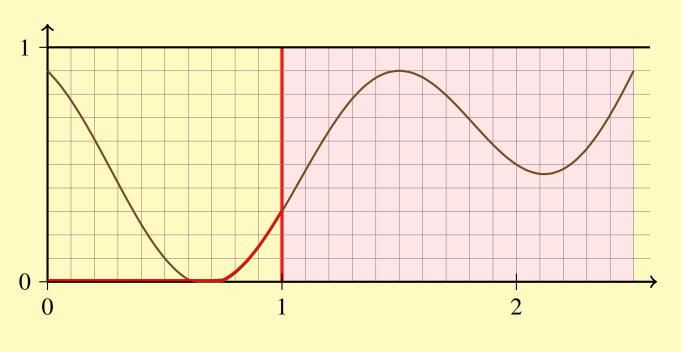

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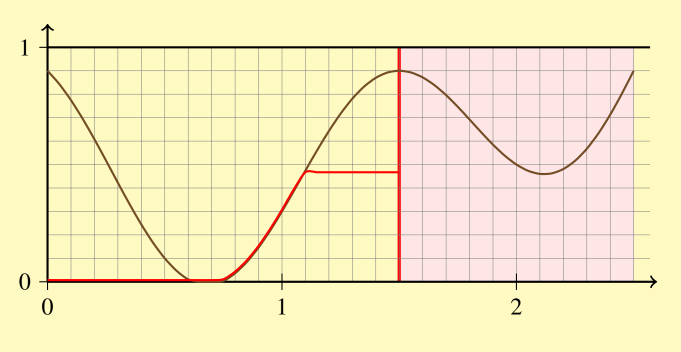

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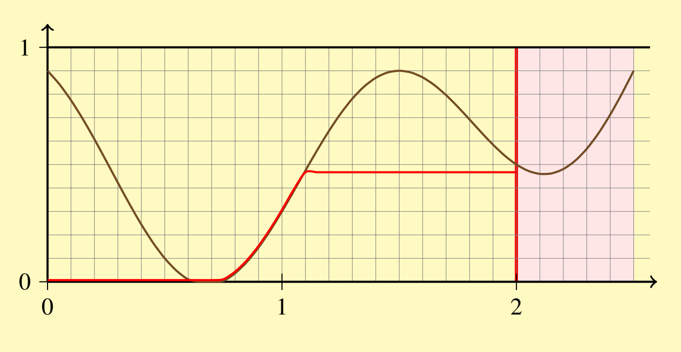

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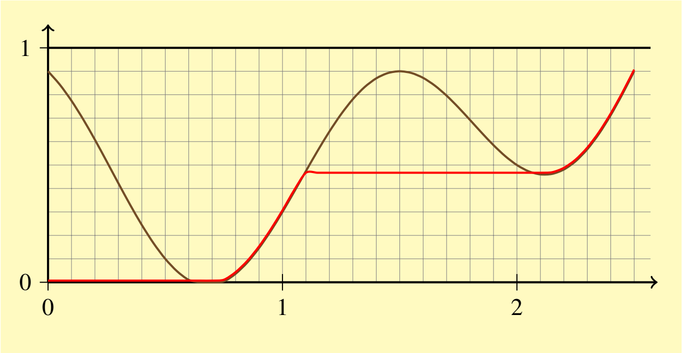

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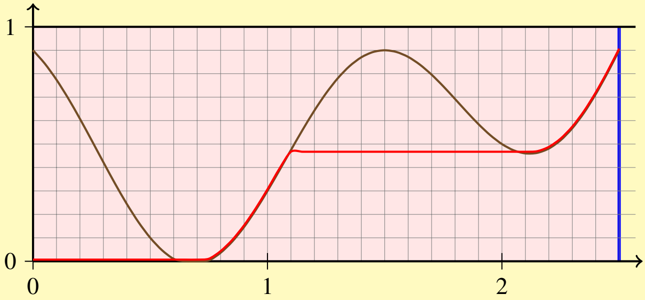

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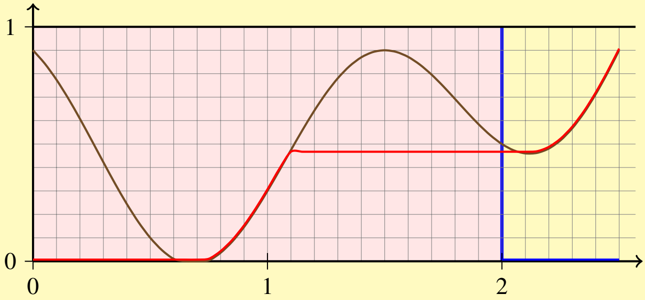

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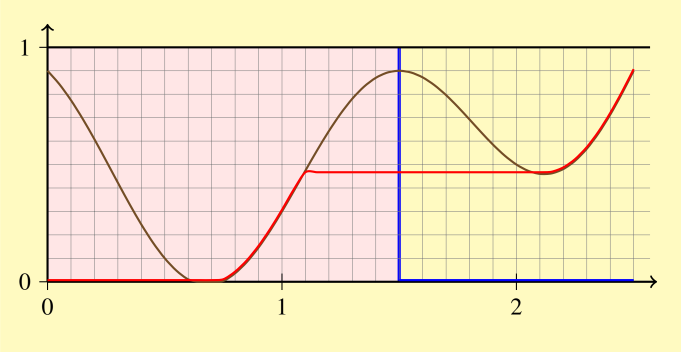

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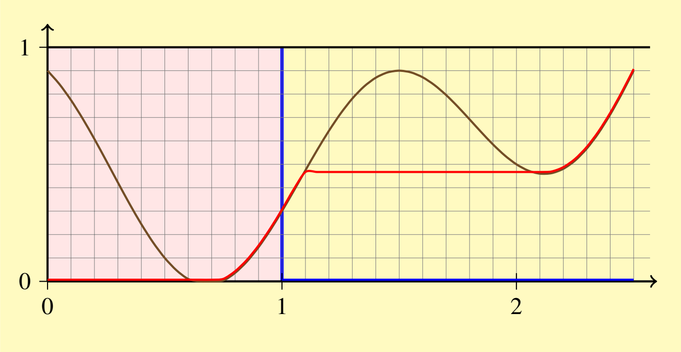

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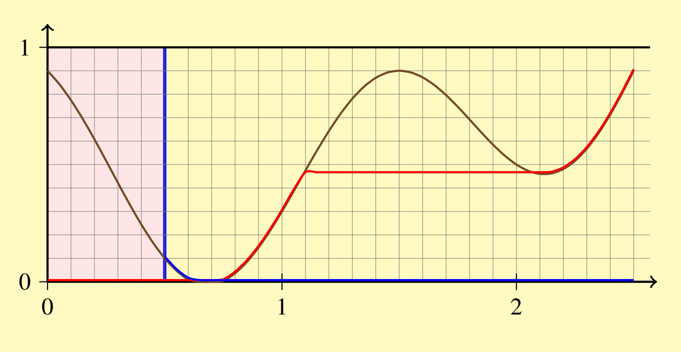

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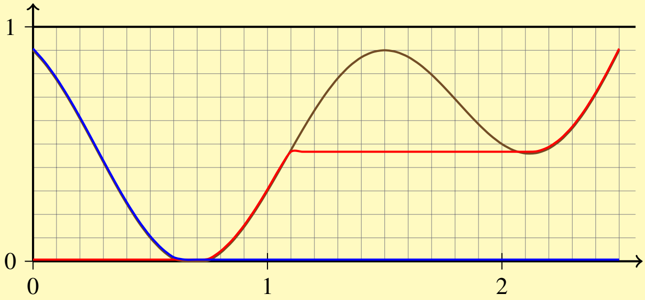

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Basic definitions - tense operators on distributive lattices

Definition

By the *tense distributive algebra* is meant an algebra $(A; \vee, \wedge, 0, 1, G, P, H, F)$ such that $\mathbf{A} = (A; \vee, \wedge, 0, 1)$ is a bounded distributive lattice with an induced order $\leq$, (P, G) and (F, G) are Galois connections on A such that, for all $p, q \in A$,

$$G(p) \leq G(0) \vee F(p) \quad \text{and} \quad F(1) \wedge G(p) \leq F(p),$$

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G, P, H and F are called *tense operators* on the tense distributive algebra.

Let $(\mathbf{A}_1; G_1, P_1, H_1, F_1)$ and $(\mathbf{A}_2; G_2, P_2, H_2, F_2)$ be tense distributive algebras. A *morphism of tense distributive algebras* is a morphism of distributive lattices $f : A_1 \to A_2$ which simultaneously commutes with the respective tense operators.

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Basic results – frames on complete distributive lattices

Theorem

Let $\mathbf{M}$ *be a complete distributive lattice,* (T,R) *be a time frame,* $\widehat{G},\widehat{P},\widehat{H}$ *and* $\widehat{F}$ *be maps from* M^T *into* M^T *defined by*

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$$((\forall s \in T)\, s(a) \leq s(b)) \implies a \leq b$$

for any elements $a, b \in P$.

Recall that, for any bounded distributive lattice $\mathbf{A} = (A; \vee, \wedge, 0, 1)$, we have a full set $T_{\mathbf{A}}^{\text{dist}}$ of morphisms of bounded lattices into the two-element bounded distributive lattice $\mathbf{2} = (\{0,1\}; \vee, \wedge, 0, 1)$.

The following result is well known.

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The following result is well known.

Proposition

Let $\mathbf{A} = (A;\vee,\wedge,0,1)$ *be a distributive lattice. Then the map* $i_{\mathbf{A}} : A \to 2^{T_{\mathbf{A}}^{dist}}$ *given by* $i_{\mathbf{A}}(a)(s) = s(a)$ *for all* $a \in A$ *and all* $s \in T_{\mathbf{A}}^{dist}$ *is an order reflecting morphism of bounded lattices such that* $i_{\mathbf{A}}(A)$ *is a sub-distributive lattice of* $\mathbf{2}^{T_{\mathbf{A}}^{dist}}$.

Outline

The representation theorem for semi-lattice morphisms

Theorem

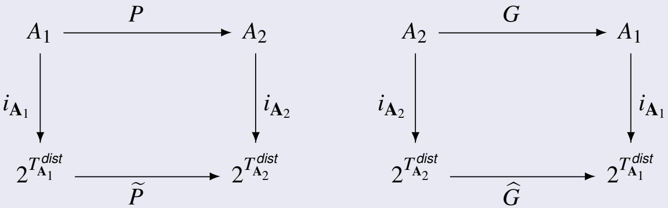
Let $\mathbf{A}_1 = (A_1; \vee, \wedge, 0, 1)$ *and* $\mathbf{A}_2 = (A_2; \vee, \wedge, 0, 1)$ *be bounded distributive lattices,* $P : A_1 \to A_2$ *and* $G : A_2 \to A_1$ *mappings such that* P *preserves finite joins and* G *preserves finite meets. Let us put*

$$R_G = \{(s,t) \in T_{\mathbf{A}_1}^{\mathit{dist}} \times T_{\mathbf{A}_2}^{\mathit{dist}} \mid (\forall b \in A_2)(s(G(b)) \leq t(b))\}$$

and

$$R^P = \{(s,t) \in T_{\mathbf{A}_1}^{\mathit{dist}} \times T_{\mathbf{A}_2}^{\mathit{dist}} \mid (\forall a \in A_1)(t(P(a)) \geq s(a))\}.$$

Then the maps $i_{\mathbf{A}_1}$ *and* $i_{\mathbf{A}_2}$ *are order reflecting morphisms of bounded lattices into the complete bounded distributive lattices* $\mathbf{2}^{T_{\mathbf{A}_1}^{\mathit{dist}}}$ *and* $\mathbf{2}^{T_{\mathbf{A}_2}^{\mathit{dist}}}$ *such that the following diagrams commute:*



The representation theorem for Galois connections

$$\begin{array}{ccc} A_1 & \xrightarrow{\quad P \quad} & A_2 \\ {\scriptstyle i_{\mathbf{A}_1}}\big\downarrow & & \big\downarrow{\scriptstyle i_{\mathbf{A}_2}} \\ 2^{T^{\text{dist}}_{\mathbf{A}_1}} & \xrightarrow[\quad \widetilde{P} \quad]{} & 2^{T^{\text{dist}}_{\mathbf{A}_2}} \end{array} \qquad \begin{array}{ccc} A_2 & \xrightarrow{\quad G \quad} & A_1 \\ {\scriptstyle i_{\mathbf{A}_2}}\big\downarrow & & \big\downarrow{\scriptstyle i_{\mathbf{A}_1}} \\ 2^{T^{\text{dist}}_{\mathbf{A}_2}} & \xrightarrow[\quad \widehat{G} \quad]{} & 2^{T^{\text{dist}}_{\mathbf{A}_1}} \end{array}$$

where $\widehat{G}(q)(s) = \bigwedge\{q(t) \mid sR_G t\}$ and $\widetilde{P}(m)(t) = \bigvee\{m(s) \mid sR^P t\}$ for all $q \in 2^{T^{\text{dist}}_{\mathbf{A}_2}}$ and $m \in 2^{T^{\text{dist}}_{\mathbf{A}_1}}$.

In particular, if (P, G) is a Galois connection then $R_G = R^P$, $\widetilde{P} = \widehat{P}$, $\widetilde{G} = \widehat{G}$ and $(\widetilde{P}, \widehat{G})$ is a Galois connection, where $\widetilde{G}(q)(s) = \bigwedge\{q(t) \mid sR^P t\}$ and $\widehat{P}(m)(t) = \bigvee\{m(s) \mid sR_G t\}$ for all $q \in 2^{T^{\text{dist}}_{\mathbf{A}_2}}$ and $m \in 2^{T^{\text{dist}}_{\mathbf{A}_1}}$.

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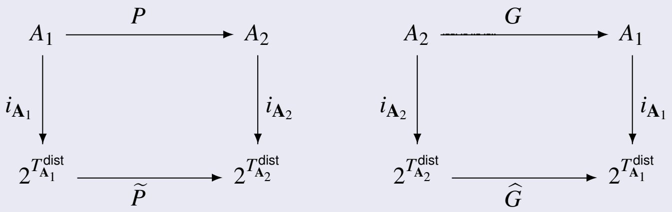

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In particular, if (P, G) is a Galois connection then $R_G = R^P$, $\widetilde{P} = \widehat{P}$, $\widetilde{G} = \widehat{G}$ and $(\widetilde{P}, \widehat{G})$ is a Galois connection, where $\widetilde{G}(q)(s) = \bigwedge\{q(t) \mid sR^P t\}$ and $\widehat{P}(m)(t) = \bigvee\{m(s) \mid sR_G t\}$ for all $q \in 2^{T^{\text{dist}}_{\mathbf{A}_2}}$ and $m \in 2^{T^{\text{dist}}_{\mathbf{A}_1}}$.

The representation theorem for Galois connections

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\quad P \quad} & A_2 \\
\Big\downarrow i_{\mathbf{A}_1} & & \Big\downarrow i_{\mathbf{A}_2} \\
2^{T^{\text{dist}}_{\mathbf{A}_1}} & \xrightarrow[\quad \widetilde{P} \quad]{} & 2^{T^{\text{dist}}_{\mathbf{A}_2}}
\end{array}
\qquad\qquad
\begin{array}{ccc}
A_2 & \xrightarrow{\quad G \quad} & A_1 \\
\Big\downarrow i_{\mathbf{A}_2} & & \Big\downarrow i_{\mathbf{A}_1} \\
2^{T^{\text{dist}}_{\mathbf{A}_2}} & \xrightarrow[\quad \widehat{G} \quad]{} & 2^{T^{\text{dist}}_{\mathbf{A}_1}}
\end{array}
$$

where $\widehat{G}(q)(s) = \bigwedge\{q(t) \mid sR_G t\}$ and $\widetilde{P}(m)(t) = \bigvee\{m(s) \mid sR^P t\}$ for all $q \in 2^{T^{\text{dist}}_{\mathbf{A}_2}}$ and $m \in 2^{T^{\text{dist}}_{\mathbf{A}_1}}$.

In particular, if (P,G) is a Galois connection then $R_G = R^P$, $\widetilde{P} = \widehat{P}$, $\widetilde{G} = \widehat{G}$ and $(\widetilde{P},\widehat{G})$ is a Galois connection, where $\widetilde{G}(q)(s) = \bigwedge\{q(t) \mid sR^P t\}$ and $\widehat{P}(m)(t) = \bigvee\{m(s) \mid sR_G t\}$ for all $q \in 2^{T^{\text{dist}}_{\mathbf{A}_2}}$ and $m \in 2^{T^{\text{dist}}_{\mathbf{A}_1}}$.

Positive MN-pairs

Definition

Let $\mathbf{A}_1 = (A_1; \vee, \wedge, 0, 1)$ and $\mathbf{A}_2 = (A_2; \vee, \wedge, 0, 1)$ be bounded distributive lattices, $F, G : A_2 \to A_1$ mappings. We say that the couple (G, F) is a *positive MN-pair* if

1. P preserves finite joins,
2. G preserves finite meets,
3. for all $p_1, p_2 \in A_2$,

$$\widehat{G}(p_1 \vee p_2) \leq \widehat{G}(p_1) \vee \widehat{F}(p_2) \quad \text{and} \quad \widehat{F}(p_1) \wedge \widehat{G}(p_2) \leq \widehat{F}(p_1 \wedge p_2). \qquad \text{(MNP1)}$$

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The representation theorem for positive MN-pairs

Theorem

Let (G,F) be a positive MN-pair between bounded distributive lattices $\mathbf{A}_1$ and $\mathbf{A}_2$. Let us put

$$R_G^F = \{(s,t) \in T_{\mathbf{A}_1}^{dist} \times T_{\mathbf{A}_2}^{dist} \mid (\forall b \in A_2)(s(G(b)) \leq t(b) \text{ and } t(b) \leq s(F(b))\}.$$

Then the maps $i_{\mathbf{A}_1}$ and $i_{\mathbf{A}_2}$ are order reflecting morphisms of bounded lattices into the complete bounded distributive lattices $\mathbf{2}^{T_{\mathbf{A}_1}^{dist}}$ and $\mathbf{2}^{T_{\mathbf{A}_2}^{dist}}$ such that the following diagrams commute:

$$\begin{array}{ccc} A_2 & \xrightarrow{\quad F \quad} & A_1 \\ \downarrow{\scriptstyle i_{\mathbf{A}_2}} & & \downarrow{\scriptstyle i_{\mathbf{A}_1}} \\ 2^{T_{\mathbf{A}_2}^{dist}} & \xrightarrow[\quad \widehat{F} \quad]{} & 2^{T_{\mathbf{A}_1}^{dist}} \end{array} \qquad \begin{array}{ccc} A_2 & \xrightarrow{\quad G \quad} & A_1 \\ \downarrow{\scriptstyle i_{\mathbf{A}_2}} & & \downarrow{\scriptstyle i_{\mathbf{A}_1}} \\ 2^{T_{\mathbf{A}_2}^{dist}} & \xrightarrow[\quad \widehat{G} \quad]{} & 2^{T_{\mathbf{A}_1}^{dist}} \end{array}$$

where $\widehat{G}(q)(s) = \bigwedge\{q(t) \mid sR_G^F t\}$ and $\widehat{F}(q)(s) = \bigvee\{q(t) \mid sR_G^F t\}$ for all $q \in 2^{T_{\mathbf{A}_2}^{dist}}$.

The representation theorem for tense distributive algebras

Theorem

(Representation theorem for tense distributive algebras) *Let* $(\mathbf{A}; G, P, H, F)$ *be a tense distributive algebra,*

$$R_G = \{(s,t) \in T_{\mathbf{A}}^{dist} \times T_{\mathbf{A}}^{dist} \mid (\forall b \in A)(s(G(b)) \leq t(b))\},$$

$$R_H = \{(s,t) \in T_{\mathbf{A}}^{dist} \times T_{\mathbf{A}}^{dist} \mid (\forall b \in A)(s(H(b)) \leq t(b))\}$$

and $R_{G,H} = R_G \cap (R_H)^{-1}$. *Then the following holds.*

1. *If* $R_G = (R_H)^{-1}$ *then the map* $i_{\mathbf{A}}$ *is an order reflecting morphism of tense distributive algebras into the powerset tense distributive algebra* $(\mathbf{2}^{T_{\mathbf{A}}^{dist}}; \widehat{G}, \widehat{P}, \widehat{H}, \widehat{F})$ *given by the time frame* $(T_{\mathbf{A}}^{dist}, R_{G,H})$.
2. *If* (G,F) *and* (H,P) *are positive MN-pairs then the map* $i_{\mathbf{A}}$ *is an order reflecting morphism of tense distributive algebras into the powerset tense distributive algebra* $(\mathbf{2}^{T_{\mathbf{A}}^{dist}}; \widehat{G}, \widehat{P}, \widehat{H}, \widehat{F})$ *given by the time frame* $(T_{\mathbf{A}}^{dist}, R_{G,H})$.

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2. *If* (G, F) *and* (H, P) *are positive MN-pairs then the map* $i_{\mathbf{A}}$ *is an order reflecting morphism of tense distributive algebras into the powerset tense distributive algebra* $(\mathbf{2}^{T^{dist}_{\mathbf{A}}}; \widehat{G}, \widehat{P}, \widehat{H}, \widehat{F})$ *given by the time frame* $(T^{dist}_{\mathbf{A}}, R_{G,H})$.

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Conclusion and future work

We have developed a construction of a tense distributive algebra $(\mathbf{L}^T; \widehat{G}, \widehat{P}, \widehat{H}, \widehat{F})$ by means of a time frame (T, R) and a finite distributive lattice $\mathbf{L}$ such that the pairs $(\widehat{G}, \widehat{F})$ and $(\widehat{H}, \widehat{P})$ are positive MN-pairs.

Conversely, given a tense distributive algebra $(\mathbf{A}; G, P, H, F)$ such that (G, F) and (H, P) are positive MN-pairs we can embed it into the powerset tense distributive algebra $(\mathbf{2}^{T_{\mathbf{A}}^{\text{dist}}}; \widehat{G}, \widehat{P}, \widehat{H}, \widehat{F})$ constructed by means of the time frame $(T_{\mathbf{A}}^{\text{dist}}, R_{G,H})$.

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Other approaches and future work

Dyckhoff and Sadrzadeh obtained similar results for positive logics with adjoint modalities P_A and G_A, $A \in \mathscr{A}$ where $\mathscr{A}$ is a set of agents and our adjunction (P_A, G_A) has the following interpretation: $P_A(m)$ has to be interpreted as "agent A's uncertainty about a proposition m" and $G_A(m)$ has to be interpreted as "agent A's uncertainty about a proposition m".

To obtain a representation of their positive logics with adjoint modalities, they use the so-called *multi-modal Kripke frame* which is a tuple $(W, \leq, (R_A)_{A\in\mathscr{A}}, (R_A^{-1})_{A\in\mathscr{A}})$ where W is a non-empty set, $\leq$ is a partial order on W, each R_A is a binary relation on W and R_A^{-1} is its inverse such that

$$\leq \circ R_A^{-1} \circ \leq \subseteq R_A^{-1} \quad \text{and} \quad \geq \circ R_A \circ \geq \subseteq R_A .$$

In our case, we have only two agents, say 1 and 2, and we put, due to Representation theorem for tense distributive algebras, $W = T_{\mathbf{A}}^{\text{dist}}$, $R_1 = R_{G,H}$ and $R_2 = R_{H,G} = R_{G,H}^{-1}$ and the partial order on our set W is trivial, i.e., the identity relation. Our machinery is based on the paper of of Dunn, that is by using MN-positive pairs to build our time frame. Hence we can forget about the partial order.

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Thank you for your attention.