Orthogonal relational systems

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Joint work with I.Chajda and A.Ledda

Outline

- Orthogonal relational systems and associated orthogroupoids

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- Decomposition of a variety of orthogroupoids
- Amalgamation property

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Motivation: generalising the theory of partially ordered sets and pre-ordered sets.

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Upper cone

$U_R(a, b) = \{x \in A | (a, x) \in R \text{ and } (b, x) \in R\}$ is the *upper cone* of $a, b \in A$ (with respect to R).

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- For every $x \perp y$, $x \neq 0 \neq y$ there exists a supremal element x, y in $U_R(x, y)$.

From relational systems to groupoids

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Induced groupoids

Let $\mathbf{A} = \langle A, R \rangle$ a relational system. We define the *induced* groupoid $\mathbf{G}(A) = \langle A, + \rangle$

- if $(x, y) \in R$ then $x + y = y$;
- if $(x, y) \notin R$ and $(y, x) \in R$ then $x + y = x$;
- if $(x, y) \notin R$ and $(y, x) \notin R$ then $x + y = y + x \in U_R(x, y)$.

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Obs: the groupoid induced by a relational system is not unique!!

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(a) $x'' = x$;

(b) $0 + x = x$ and $1 + x = 1$, where $0 = 1'$;

(c) $x + x' = 1$;

(d) if $x + z = z$ and $x' + z = z$ then $z = 1$;

(e) $(((z + y)' + (z + x))' + (z + y)') + z' = z'$;

(f) $x + (x + y) = x + y$ and $y + (x + y) = x + y$.

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Any orthogroupoid induces a relational system $\mathbf{A}(G) = \langle G, R_G\rangle$, where $(a, b) \in R_G$ if and only if $a + b = b$.

Some properties

Proposition

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Theorem 1

Let $\mathbf{D} = \langle D, +, ', 1\rangle$ be an orthogroupoid and R the induced relation. Then the relational system $\mathbf{A}(D) = \langle D, R, ', 1\rangle$ is orthogonal and R is reflexive.

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Theorem 2

Let $\mathbf{A} = \langle A, R,', 1\rangle$ be an orthogonal relational system with R reflexive and transitive. Then any induced groupoid is an orthogroupoid.

Church algebras

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In a Church algebra the following operations are definable

$$x \wedge y = q(x, y, 0)$$
$$x \vee y = q(x, 1, y)$$
$$x^* = q(x, 0, 1)$$

Central elements

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Theorem (Salibra, Ledda, Paoli and Kowalski)

Let $\mathbf{A}$ a Church algebra. Then $\mathrm{Ce}(\mathbf{A}) = \langle \mathrm{Ce}(A), \wedge, \vee, ^*, 0, 1 \rangle$ is a Boolean algebra isomorphic to the Boolean algebra of factor congruences of $\mathbf{A}$.

Church algebras and orthogroupoids

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The variety of 0-commutative orthogroupoid is a Church variety, with witnessing term $q(x, y, z) = (x + z) \cdot (x' + y)$.

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Proposition

Let $\mathbf{A}$ be 0-commutative orthogroupoid and $\mathrm{Ce}(A)$ the set of central elements. Then $\mathrm{Ce}(\mathbf{A}) = \langle \mathrm{Ce}(A), +, \cdot, ', 0, 1 \rangle$ is a Boolean algebra.

A decomposition theorem

Let **A** be a Church algebra of type ν, $e \in A$ a central element, it is possible to define $\mathbf{A}_e = (A_e; g_e)_{g \in \nu}$:

$$A_e = \{e \wedge b : b \in A\}; \quad g_e(e \wedge \overline{b}) = e \wedge g(e \wedge \overline{b}).$$

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Theorem

Let **A** a 0-commutative orthogroupoid s.t. $\mathrm{Ce}(\mathbf{A})$ is a Boolean algebra with a denumerable number of atoms. Then

$$\mathbf{A} = \prod_{e \in At(\mathbf{A})} \mathbf{A}_e$$

is a decomposition as product of directly indecomposable algebras.

Amalgamation property

A V-formation is a 5-tuple $(\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, i, j)$, where $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2$ are similar algebras and i, j embeddings.

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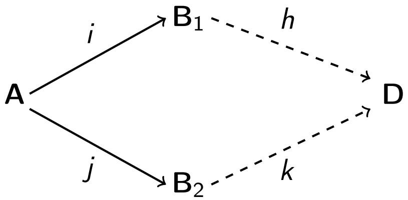
A class $\mathcal{K}$ of similar algebras has the amalgamation property, if for any V-formation there exists an algebra $\mathbf{D} \in \mathcal{K}$ e embeddings $h : \mathbf{B}_1 \to \mathbf{D}, k : \mathbf{B}_2 \to \mathbf{D}$ s.t. $k \circ j = h \circ i$



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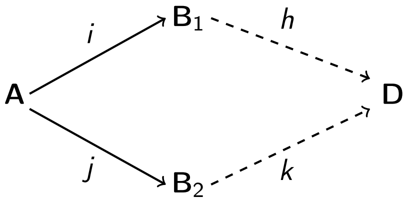

In particular $\mathcal{K}$ ha the strong amalgamation property if k and h can be taken s.t. $k \circ j\,(\mathbf{A}) = h\,(\mathbf{B}_1) \cap k\,(\mathbf{B}_2)$.

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Sketch of the proof

Given a V-formation, let $D = B_1 \cup B_2$, $x'^D = x'^{B_i}$ and

$$x \oplus y = \begin{cases} x +^{B_i} y, & \text{if } x, y \in B_i; \\ 1, & \text{otherwise.} \end{cases}$$

$\mathbf{D} = \langle D, \oplus, ', 1 \rangle$ is an orthogroupoid.
Assuming with no loss of generality $B_1 \cap B_2 = A$ one gets the conclusion.

The end!

Thanks for your attention!!