Coherence for Categories of Posets with Applications

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[1] Thanks to Alex Kurz and Peter Jipsen for stimulating this line of work.

Motivation

- Partially ordered sets are the basic structures of algebraic logic:
  - A set (of "propositions")
  - An "entailment" relation between them: $p \Rightarrow q$
- Additional logical structure: connectives with rules.
- Want to put this onto a category-theoretic footing.

Order Enriched Categories

Definition

Order Enriched Category: a category in which hom sets are partially ordered and composition is monotonic in both arguments.

Examples

- Pos itself
- Any category that is concrete over Pos
- Rel – morphisms ordered by $\subseteq$
- Pos* – posets with weakening relations: $R\colon A \nrightarrow B$ s.t.

$$a \leq a' \; R \; b' \leq b \text{ implies } a \; R \; b$$

- DLat* – bounded dist. lattices with weakening relations $R\colon A \nrightarrow B$ that are also sublattices of $A \times B$

Map-like Behavior of Weakening Relations

Between posets, two weakening relations arise naturally from a monotonic function.

For $f\colon A \to B$, define

- $\hat{f}\colon A \nrightarrow B$ by $a\,\hat{f}\,b$ iff $f(a) \le b$
- $\check{f}\colon B \nrightarrow A$ by $b\,\check{f}\,a$ iff $b \le f(a)$.

Lemma

For any monotonic function $f\colon A \to B$,

$$id_B \le \check{f};\hat{f} \text{ and } \hat{f};\check{f} \le id_A$$

Definition

In any poset enriched category $\mathcal{A}$,

- A map is a morphism with a lower adjoint.
- $\mathrm{Map}(\mathcal{A})$ is the subcategory of maps.

From map-like behavior to honest functions

Lemma

The categories $Map(Pos^*)$ and Pos are equivalent.

Proof.

For function $f\colon A \to B$, we have $\hat{f}$ adjoint to $\check{f}$.
For an adjoint pair of weakening relations (R^*, R_*), define

$$f_m(a) = b \text{ iff } a\ R^*\ b\ R_*\ a.$$

□

Note: An analogous fact is true for

- DLat* and DLat
- Set* (also known as Rel) and Set (discrete partial orders)
- many others.

Cartesian Bicategories

Definition (Carboni & Walter)

A cartesian bicategory is

- Poset enriched
- Symmetric monoidal: $\otimes$, $\mathbb{I}$ with the usual natural isos
- $\otimes$ is monotonic on hom sets
- every object is equipped with a comonoid:
  - $\hat{\delta}_A \colon A \to A \otimes A$
  - $\hat{\kappa}_A \colon A \to \mathbb{I}$
- all morphims are lax homomorphisms for the comonoid:

$$R; \hat{\delta}_B \leq \hat{\delta}_A; (R \otimes R)$$
$$R; \hat{\kappa}_B \leq \hat{\kappa}_A$$

- $\hat{\delta}_A$ and $\hat{\kappa}_A$ are maps [they have adjoints].
- $\hat{\delta}_A; \check{\delta}_A = \mathrm{id}_A$

Pos*, Lat*, DLat*, BA* and Set* are cartesian bicategories

In Pos*

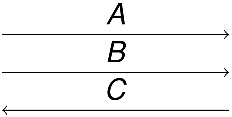
- Cartesian product $A \otimes B$ and $\mathbb{I} = \{\star\}$ yield the symmetric monoidal structure in all cases.
- The relations
  - $a\,\hat{\delta}_A\,(b, c)$ if and only if $a \leq b$ and $a \leq c$
  - $a\,\hat{\kappa}\,\star$ (all a)

  determine cartesian bicategory structure ($\leq$ is equality in Set*)
- Also Pos* is *compact closed*: The order dual A^∂ of a poset is again such an object. One has to check that these are duals in the correct sense.
- In Set*, $A^\partial = A$.
- In Lat*, DLat* and BA*, same as in Pos*.

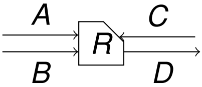
String Diagrams For Symmetric Monoidal Categories

Symmetric monoidal (and compact closed) categories have a coherence theorem based on string diagrams

A diagram of $A \otimes B \otimes C^{\partial}$:



A diagram of $R \colon A \otimes B \looparrowright C^{\partial} \otimes D$:

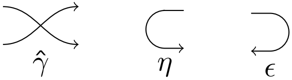


Theorem (Joyal & Street)

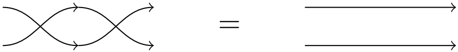
Two diagrams denote the same morphism in all compact closed categories iff they are homotopically equivalent (in $\mathbb{R}^4$).

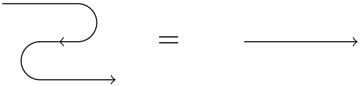
Some Details of Diagrams

Symmetry is “crossed wires”. Unit and counit are “u-turns”.



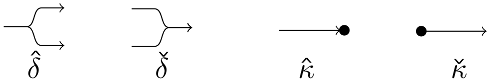
So the compact closed structure is reflected in various equations:



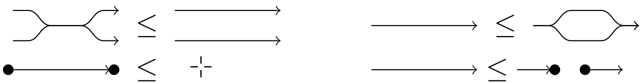


Bicartesian Enrichment

Diagrams for the diagonals



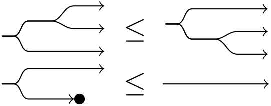
Map axioms



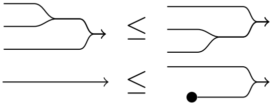
More Axioms (and Lemmas)

Comonoid/monoid

Comonoid Axioms



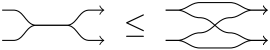
Monoid lemmas



Split monicity axiom for δ

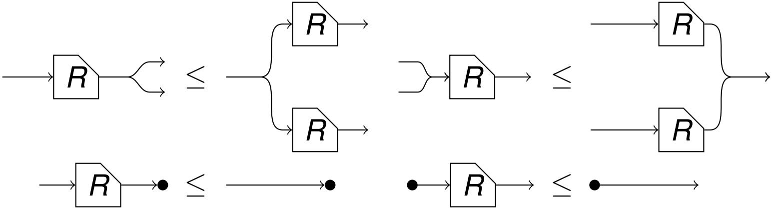
Lax Naturality Axioms and lemmas

Weak Frobenius Axiom (laxity for $\check{\delta}$ wrt $\hat{\delta}$)



Laxity for basic morphisms

Axioms Lemmas



Coherence Theorems

Theorem

Let $\leq$ be the least pre-order on string diagrams including the axioms and closed under composition and $\otimes$ (stacking). Then the poset reflection of $\leq$ determines an initial cartesian bicategory (for a given set of basic objects and morphisms).

Theorem

The same construction works for compact closed cartesian bicategories.

Theorem

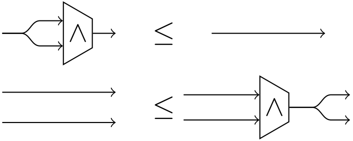
The same construction also works when $\leq$ is augmented with an inequational theory (a set of pairs of diagrams).

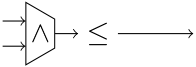
Lattice-like Objects in Cartesian Bicategories

Meets and joins

- And object is meet semilattice-like if $\hat{\delta}_A$ is a comap (it is already being a map).

  So there is a morphism  satisfying

  It is easy to show that $\wedge$ is idempotent, associative and commutative and deflating: 

- Dually, A is join semilattice-like if $\check{\delta}_A$ is a map.

More on Lattices

Lemma

In Pos:

- *A poset P is an actual meet semilattice iff it is meet semilattice-like.*
- *A poset P is an actual join semilattice iff it is join semilattice-like.*

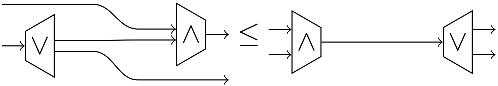
Moreover

- Boundedness is characterized by $\hat{\kappa}_A$ being a comap ($\top$) or $\check{\kappa}_A$ being a map ($\bot$).
- What about distributivity?

Distributivity

Lemma

A lattice-like object in a cartesian bicategory is distributive (i.e., $\wedge$ distributes over $\check{\delta}$) if and only if

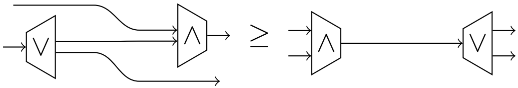


The proof is entirely "stringy". That is, we can use only the string rewriting in the initial bicartesian category of string diagrams.

Complementedness

Lemma

In Pos**, if an object is a distributive lattice, then it is complemented if and only if*



Remark

- This condition is dual to the Frobenius Law (FL) for the bialgebra $(\hat{\delta}, \check{\delta}, \hat{\kappa}, \check{\kappa})$.
- If FL holds for all objects, the bicartesian category is a regular allegory.
- “Complemented distributive lattice” is dual to “discrete”. [I do not yet know how to make this precise.]

Other Examples and Consreuctions

Examples

- Compact pospaces (Gehrke mentioned them yesterday) by taking *closed weakening relations* as morphisms. Then maps are bijective with continuous monotonic functions.
- Proximity lattices (not quite discussed yesterday).
- Rel – all objects satisfy Frobenius Law

Constructions

- Map-comma: Objects are maps into a base B. Morphisms are lax homomorphisms.
- T-algebras for a lax monad on $\mathcal{A}$ (?).
- Karoubi
- $(\mathsf{Pos}^*)^{\mathcal{A}^{\mathsf{op}}}$ – "presheaves" over Pos^*.

Thanks