De Vries Powers: A generalization of Boolean powers for compact Hausdorff spaces

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Boolean powers of commutative rings (including $\mathbb{Z}$) appeared in work of Conrad and Ribenboim in studying ℓ-groups, and in work of Bergman and Rota. We have axiomatized them for an arbitrary base ring.

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There may be other idempotents of $C(X, A)$ beyond these characteristic functions. For example, if $e \neq 0, 1$ is idempotent in A, then the constant function $x \mapsto e$ is idempotent in $C(X, A)$.

Specker A-algebras

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Theorem. Let S be a commutative A-algebra. Then S is a Boolean power of A iff it is a Specker A-algebra.

Examples

- If X is a Stone space, then the $\mathbb{R}$-subalgebra of $C(X, \mathbb{R})$ generated by the continuous characteristic functions is a Specker $\mathbb{R}$-algebra and is $C(X, \mathbb{R}_{disc})$.

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- If X is a Stone space, then the $\mathbb{R}$-subalgebra of $C(X, \mathbb{R})$ generated by the continuous characteristic functions is a Specker $\mathbb{R}$-algebra and is $C(X, \mathbb{R}_{disc})$.
- If F is a field and S is a commutative F-algebra, then the F-subalgebra of S generated by $\mathrm{Id}(S)$ is a Specker F-algebra.
- More generally, if S is a torsion free A-algebra with A an integral domain, then the subalgebra of S generated by $\mathrm{Id}(S)$ is a Specker A-algebra.

Moving to Compact Hausdorff Spaces

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There are several dualities involving compact Hausdorff spaces. We found that de Vries duality was particularly appropriate for us.

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It has a canonical proximity $U \prec V$ iff $\mathrm{Cl}(U) \subseteq V$.

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6. $a \prec b$ implies there is $c \in B$ such that $a \prec c \prec b$.
7. $a \neq 0$ implies there is $0 \neq b \in B$ such that $b \prec a$.

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One functor sends a space X to $(\mathrm{RO}(X), \prec)$. Going backwards is accomplished by defining end filters (or ideals) of a de Vries algebra, topologizing the set of ends, and seeing that the result is a compact Hausdorff space.

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We "recover" Stone duality by recognizing, for a Stone space X, clopen subsets U are characterized by $U \prec U$. Thus, $\{U \in \mathrm{RO}(X) : U \prec U\}$ is the dual Boolean algebra to X.

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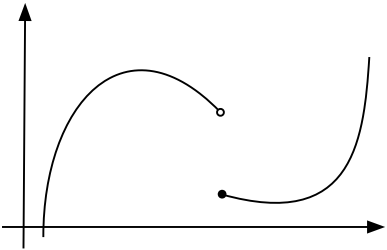
In Boolean powers, characteristic functions of clopen sets play an important role. Given de Vries duality, it is reasonable to consider characteristic functions of regular opens.

Normal Functions

Dilworth described the MacNeille completion of $C(X, \mathbb{R})$ in terms of normal functions. If f is bounded, let f^* be the smallest upper semicontinuous function above f and f_* the largest lower semicontinuous function below f. Then f is **normal** if $f = (f^*)_*$.



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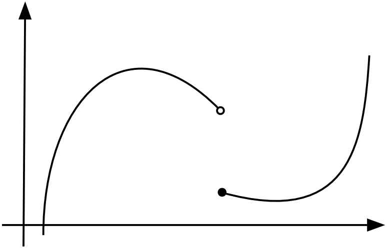

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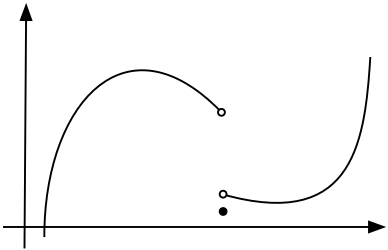

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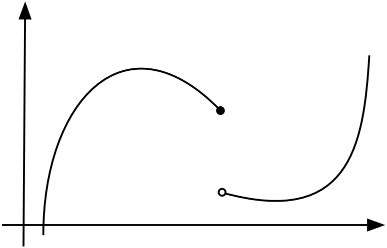

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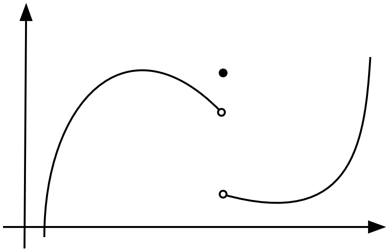

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We call a finitely valued $f : X \to A$ **normal** if $f^{-1}(\uparrow a)$ is regular open for each $a \in A$.

Let $FN(X, A)$ be the set of all finitely valued normal functions from X to A.

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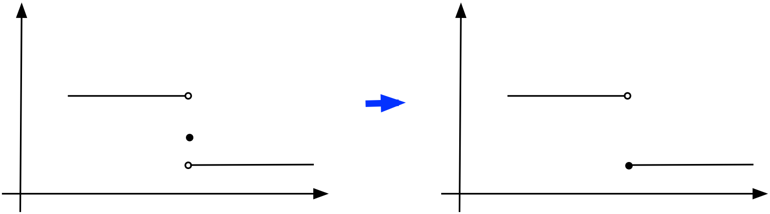

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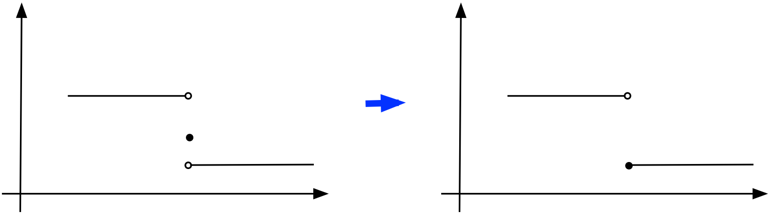

If U is a subset of X, then the normalization of χ_U is $\chi_{\mathrm{Int}(\mathrm{Cl}(U))}$.

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In particular, we can define $+$, $\cdot$, and $-$ on $FN(X, A)$ and get a commutative f-ring.

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7. $f > 0$ implies there is $0 < g$ with $g \prec f$.

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$$\{f \in FN(X, A) : f \prec f\}.$$
- $FN(X, A)$ is a Specker A-algebra.
- Idempotents of $FN(X, A)$ correspond to regular open subsets of X.
- It follows that idempotents form a complete Boolean algebra. Therefore, $FN(X, A)$ is a **Baer** ring.

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Proximity Specker A-algebras

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Let S be a torsion-free f-algebra over A. Then $\prec$ is a **proximity** if it satisfies the axioms of the relation on $FN(X, A)$ we defined above.

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The previous slide motivates the following definitions.

Let S be a torsion-free f-algebra over A. Then $\prec$ is a **proximity** if it satisfies the axioms of the relation on $FN(X, A)$ we defined above.

A pair $(S, \prec)$ is a **proximity A-algebra** if S is a torsion-free f-algebra over A and $\prec$ is a proximity on S.

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If X is compact Hausdorff, then $(FN(X, A), \prec)$ is a proximity Baer Specker A-algebra.

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Another way is to send S to the space of minimal prime ideals with the Zariski topology.

A third way is to introduce end ideals. The three approaches yield the same space.

Theorem. There is a category equivalence between compact Hausdorff spaces and proximity Baer Specker A-algebras.

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As Boolean powers generalize Stone duality, de Vries powers generalize de Vries duality.

Recall the Katětov-Tong theorem: If f, g are bounded real-valued functions on X with $f^* \leq g_*$, then there is a continuous h with $f \leq h \leq g$.

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If $f, g \in FN(X, \mathbb{R})$ with $f \prec g$, then $f^* \leq g_*$. However, a continuous h with $f \leq h \leq g$ need not be finitely valued.

Thus, the in between axiom doesn't follow from Katětov-Tong, so we need another approach.

It is natural to write elements as linear combinations of pairwise disjoint idempotents, this is not convenient for proximities. We instead use the notion of a **decreasing decomposition**, which is related to Mundici's good sequences.

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Let S be a Specker A-algebra. If $s \in S$, then we may write $s = a_0 + a_1 e_1 + \cdots + a_n e_n$ with $a_i \geq 0$ if $i \geq 1$ and $e_1 \geq e_2 \geq \cdots \geq e_n \in \mathrm{Id}(S)$.

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This representation is well behaved with respect to normalization. It is also convenient in working with proximities.

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Thanks for listening, and thanks to the organizers for their work setting up this conference.