Proof by Order

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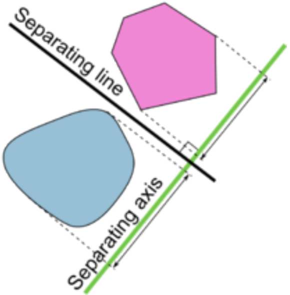
A Theorem of the Alternative

Theorem (Gordan 1873)

Given $M \in \mathbb{R}^{m \times n}$, exactly one of the following systems has a solution:

(a) $y^T M > 0$ *for some* $y \in \mathbb{R}^m$

(b) $Mx = 0,\ x \geq 0,\ x \neq 0$ *for some* $x \in \mathbb{R}^n$.



This Talk

How do **proof** and **order** interact in lattice-ordered groups?

Partially Ordered Groups

A **partially ordered group** (or **po-group**) consists of a group **G** equipped with a partial order $\leq$ satisfying for all $a, b, c \in G$,

$$a \leq b \quad \Longrightarrow \quad ac \leq bc \quad \text{and} \quad ca \leq cb.$$

A partially ordered group **G** is

- an **ordered group** (or **o-group**) if $\leq$ is total
- a **lattice-ordered group** (or **ℓ-group**) if $\leq$ is a lattice order.

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Lattice-Ordered Groups

A **lattice-ordered group** (or **ℓ-group**) is also an algebraic structure

$$\mathbf{L} = (L, \wedge, \vee, \cdot, ^{-1}, e)$$

satisfying the following conditions:

- $(L, \cdot, ^{-1}, e)$ is a group
- $(L, \wedge, \vee)$ is a lattice (with $a \leq b \;\Leftrightarrow\; a \wedge b = a$)
- $a(b \vee c)d = abd \vee acd$ for all $a, b, c, d \in L$.

It follows also that $\mathbf{L}$ is distributive and satisfies $e \leq a \vee a^{-1}$.

Syntax

Let us call a variable x and its inverse x^{-1} **literals**, and consider terms built from literals and operation symbols e, $\wedge$, $\vee$, and $\cdot$.

We also define inductively an **inverse operation**:

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\begin{array}{rclcrcl}
\overline{e} & = & e & \qquad & \overline{t_1 \cdot t_2} & = & \overline{t_2} \cdot \overline{t_1} \\
\overline{x} & = & x^{-1} & & \overline{t_1 \wedge t_2} & = & \overline{t_1} \vee \overline{t_2} \\
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\end{array}
$$

For any term t, there exist I, J_i ($i \in I$) and **group terms** t_{ij} such that in any ℓ-group $\mathbf{G}$,

$$\mathbf{G} \models t \approx \bigwedge_{i \in I} \bigvee_{j \in J_i} t_{ij}.$$

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Rewriting Equations

Moreover, for ℓ-group terms s, t and any ℓ-group $\mathbf{G}$,

$$\mathbf{G} \models s \approx t \qquad \Longleftrightarrow \qquad \mathbf{G} \models e \leq (\bar{s} \cdot t) \wedge (\bar{t} \cdot s).$$

So to check the validity of ℓ-group equations, it suffices to check the validity of equations $e \leq t$ where t is a *join of group terms*.

Abelian ℓ-Groups

The **integers** provide an important example of an ℓ-group:

$$\mathbf{Z} = (\mathbb{Z}, \min, \max, +, -, 0).$$

Indeed, this algebra generates the variety A of **abelian ℓ-groups.**

A Theorem of the Alternative Revisited

Theorem

The following are equivalent for group terms $t_1, \ldots, t_n$:

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Interpreted in **Z**, this is (almost)...

Theorem (Gordan 1873)

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Positive Cones

The **positive cone** $P = \{a \in G : a \geq e\}$ of a po-group **G** satisfies

(i) $PP \subseteq P$

(ii) $P \cap \overline{P} = \{e\}$,

and if **G** is an o-group, also

(iii) $G = P \cup \overline{P}$.

Conversely, if a subset P of an abelian group **G** satisfies (i)-(ii), then **G** is partially ordered (totally ordered if P also satisfies (iii)) by

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Proof.

(2)$\Rightarrow$(1) Easy, using the fact that $\mathrm{A} \models e \leq st$ implies $\mathrm{A} \models e \leq s \vee t$.

(1)$\Rightarrow$(2) Let P be the subsemigroup of the free ω-generated abelian group **F** generated by $e, \overline{t_1}, \ldots, \overline{t_n}$. If (2) fails, P defines a partial order on **F**, which, as **F** is torsion-free, extends to a total order on **F**. Then $e \leq t_1 \vee \ldots \vee t_n$ fails in the o-group **F**, so $\mathrm{A} \not\models e \leq t_1 \vee \ldots \vee t_n$. □

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Sequents and Hypersequents

A **sequent** Γ is a finite sequence of literals $t_1, \ldots, t_n$ with inverse

$$\overline{t_1, \ldots, t_n} = \overline{t_n}, \ldots, \overline{t_1},$$

interpreted as the group term $t_1 \cdot \ldots \cdot t_n$ for $n > 0$, and as e for $n = 0$.

A **hypersequent** is a finite set of sequents, written

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A Hypersequent Calculus for Abelian ℓ-Groups

$\mathbf{A} \models e \leq \mathcal{G} \quad \Longleftrightarrow \quad \mathcal{G}$ is derivable using the rules

$$\frac{}{\Delta, \overline{\Delta}}\ (\text{ID}) \qquad \frac{\Pi, \Delta, \Gamma, \Sigma}{\Pi, \Gamma, \Delta, \Sigma}\ (\text{EX}) \qquad \frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}}\ (\text{EW}) \qquad \frac{\mathcal{G} \mid \Gamma, \Delta}{\mathcal{G} \mid \Gamma \mid \Delta}\ (\text{SPLIT})$$

This is a one-sided "bare bones" version of a calculus in

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Let V be a variety of semilinear involutive commutative residuated lattices satisfying $na = a^n$ for each $n \in \mathbb{N}^+$. Then the following are equivalent for multiplicative terms $t_1, \dots, t_n$:

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(2) $V \models e \leq \lambda_1 t_1 \oplus \cdots \oplus \lambda_n t_n$ *for some* $\lambda_1, \dots, \lambda_n \in \mathbb{N}$ *not all* 0.

In particular, for the variety of **Sugihara monoids**, we obtain a version of the hypersequent calculus for the logic **R-Mingle** defined in

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But Now. . .

What happens for (possibly **non-abelian**) ℓ-groups?

Automorphism ℓ-Groups

The order-preserving bijections on a chain Ω with function composition and inverse form a group $\mathbf{Aut}(\Omega)$ lattice-ordered by

$$f \leq g \qquad \Longleftrightarrow \qquad f(a) \leq g(a) \;\text{ for all } a \in \Omega.$$

Theorem (Holland 1963)

Every ℓ-group embeds into $\mathbf{Aut}(\Omega)$ for some chain Ω.

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The variety LG of ℓ-groups is generated by $\mathbf{Aut}(\mathbb{R})$.

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Validity in ℓ-Groups (1)

Let **F** denote the **free ω-generated group**, and let $S(X)$ denote the **subsemigroup** of a group **G** generated by a set X of elements of G.

Theorem

For group terms $t_1, \ldots, t_n$, exactly one of the following holds:

(a) $\mathsf{LG} \models e \leq t_1 \vee \ldots \vee t_n$.

(b) $S(e, \overline{t_1}, \ldots, \overline{t_n})$ *extends to the positive cone of a right order on* **F**.

Proof.

If (b) holds, then **F** admits a right order whose positive cone includes $\overline{t_1}, \ldots, \overline{t_n}$, and it follows that $\mathbf{Aut}(\mathbf{F}) \not\models e \leq t_1 \vee \ldots \vee t_n$, i.e., (a) fails.

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If (b) holds, then **F** admits a right order whose positive cone includes $\overline{t_1}, \ldots, \overline{t_n}$, and it follows that **Aut**(**F**) $\not\models e \leq t_1 \vee \ldots \vee t_n$, i.e., (a) fails.

The other part can be proved algebraically or proof-theoretically... □

Validity in ℓ-Groups (1)

Let **F** denote the **free ω-generated group**, and let $S(X)$ denote the **subsemigroup** of a group **G** generated by a set X of elements of G.

Theorem

For group terms $t_1, \ldots, t_n$, exactly one of the following holds:

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Theorem (Kopytov and Medvedev 1994)

A partial right order of a group **G** *with positive cone* P *extends to a right order of* **G** *if and only if for all* $a_1, \ldots, a_n \in G \setminus \{e\}$*, there exist* $\delta_1, \ldots, \delta_n \in \{-1, 1\}$ *such that* $e \notin S(\{a_1^{\delta_1}, \ldots, a_n^{\delta_n}\} \cup (P \setminus \{e\}))$.

Let us call a group term t **valid** if $e \approx t$ is valid in all groups.

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A Hypersequent Calculus for ℓ-Groups

LG $\models$ $e \leq \mathcal{G}$ $\iff$ $\mathcal{G}$ is derivable using the rules

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