Algorithmic Correspondence and Proof Theory for Strict Implication[1]

Zhiguang Zhao
This is a joint work in process with Minghui Ma (China)

Delft University of Technology, Delft, The Netherlands

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[1] The original title was *Strict Implication Logics and Lambek Calculi*. We changed into the current title which is more precise.

Outline

1. Weak strict implication logics
2. Bounded distributive lattices with implication
3. Algorithmic correspondence theory
4. Conservativity
5. Gentzen-style Sequent Calculi

♠1. Weak strict implication logics

Strict implication $\phi \to \psi := \Box(\phi \supset \psi)$

1. Intuitionistic logic and subintuitionistic logics.
   1.1 G. Corsi. Weak logics with strict implication. *Zeitschrift für mathematische Logik u. Grundlagen d*, 33:389–406, 1987.
   1.2 K. Došen. Modal translations in K and D. In *Diamonds and Defaults*, 103–127. Kluwer Academic Publishers, 1993.
   1.3 A. Visser. A propositional logic with explicit fixed points. *Studia Logica*, 40(2):155–175, 1981.
2. The local consequence relation:
   2.1 S.Celani and R. Jansana. A closer look at some subintuitionistic logics. *Notre Dame Journal of Formal Logic* 42, 225–255, 2003.
   2.2 S. Celani and R. Jansana. Bounded distributive lattices with strict implication. *Mathematical Logic Quarterly*, 51(3):219–246, 2005.

Language and Semantics

The set of all *strict implication formulas*, also called *terms*, $\mathcal{L}_{term}$ is defined inductively by the following rule:

$$\mathcal{L}_{term} \ni \phi ::= p \mid \bot \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi),$$

where $p \in$ Prop. Define $\top := \bot \to \bot$, $\neg\phi := \phi \to \bot$, and $\phi \equiv \psi := (\phi \to \psi) \wedge (\psi \to \phi)$.

Sequent:

$$\Gamma \vdash \phi$$

where Γ is a finite (possibly empty) set of formulas.

Kripke Semantics

Frame: $\mathcal{F} = (W, R)$ where $W \neq \emptyset$ and $R \subseteq W^2$.
Model: $\mathcal{M} = (W, R, V)$ where $V : \text{Prop} \to \mathcal{P}(W)$ is arbitrary valuation. The *satisfaction relation* $\mathcal{M}, w \models \phi$:

1. $\mathcal{M}, w \models p$ iff $w \in V(p)$.
2. $\mathcal{M}, w \not\models \bot$.
3. $\mathcal{M}, w \models \phi \wedge \psi$ iff $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$.
4. $\mathcal{M}, w \models \phi \vee \psi$ iff $\mathcal{M}, w \models \phi$ or $\mathcal{M}, w \vdash \psi$.
5. $\mathcal{M}, w \models \phi \to \psi$ iff $\forall u \in W(wRu \;\&\; \mathcal{M}, u \models \phi \Rightarrow \mathcal{M}, u \models \psi)$.

Let $V(\phi) = \{w \in W \mid \mathcal{M}, w \models \phi\}$. For any set Σ of formulas, let $V(\Sigma) = \bigcap\{V(\phi) \mid \phi \in \Sigma\}$.

1. Validity $\mathcal{F} \models \Gamma \vdash \phi$: $V(\Gamma) \subseteq V(\phi)$ for any valuation V in $\mathcal{F}$.
2. Local consequence relation, $\Sigma \models^{l}_{\mathcal{K}} \phi$: for every valuation V in any frame in $\mathcal{K}$, $V(\Sigma) \subseteq V(\phi)$.

Weak Strict Implication Logics

Definition (Cenali & Jansana 2003)

A *weak strict implication logic* is a set of sequents L which contains all instances of the following axiom schemata:

$$(\mathrm{M1})\ \phi \to \psi, \phi \to \chi \vdash \phi \to (\psi \wedge \chi) \quad (\mathrm{M2})\ \phi \to \chi, \psi \to \chi \vdash (\phi \vee \psi) \to \chi$$

$$(\mathrm{Slly})\ \phi \to \psi, \psi \to \chi \vdash \phi \to \psi \quad (\mathrm{Id})\ \phi \vdash \phi$$

and is closed under the following rules:

$$\frac{\Gamma \vdash \phi}{\Gamma, \psi \vdash \phi}(w) \quad \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}(\bot\mathrm{R}) \quad \frac{\Gamma, \phi, \psi \vdash \delta}{\Gamma, \phi \wedge \psi \vdash \delta}(\wedge\mathrm{L}) \quad \frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}(\wedge\mathrm{R})$$

$$\frac{\Gamma, \phi \vdash \chi \quad \Gamma, \psi \vdash \chi}{\Gamma, \phi \vee \psi \vdash \chi}(\wedge\mathrm{L}) \quad \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi}(\vee\mathrm{R})$$

$$\frac{\phi \vdash \psi}{\emptyset \vdash \phi \to \psi}(\mathrm{DT}_0) \quad \frac{\Gamma \vdash \phi \quad \Gamma, \phi \vdash \psi}{\Gamma \vdash \psi}(\mathrm{cut})$$

Weak Strict Implication Logics

1. The minimal weak strict implication logic is denoted by wK_σ.
2. Deductive consequence relation For every set of formulas $\Phi \cup \{\phi\} \subseteq \mathcal{L}_{term}$, we say that ϕ is a *deductive consequence* of Φ in L (notation: $\Phi \vdash_{\mathsf{L}} \phi$) if there exists a finite subset $\Delta \subseteq \Phi$ such that the sequent $\Delta \vdash \phi$ is derivable in L.
3. Strong Completeness A weak strict implication logic L is said to be *strongly complete* with respect to a class of frames $\mathcal{K}$ if for every set of formulas $\Sigma \cup \{\phi\}$, $\Sigma \vdash_{\mathsf{L}} \phi$ iff $\Sigma \models^{l}_{\mathcal{K}} \phi$.

Some weak strict implication logics

Theorem (Celani and Jansana 2003)

The least weak strict implication logic wK_σ *is strongly complete with respect to the class of all frames.*

Sequent	First-order correspondent
(wD) $\neg\top \vdash \bot$	$\forall x \exists y Rxy$
(wT) $p \wedge (p \to q) \vdash q$	$\forall x Rxx$
(w4) $p \to q \vdash r \to (p \to q)$	$\forall xyz((Rxy \wedge Rxz) \supset Ryz)$
(wB) $p \vdash q \vee \neg(p \to q)$	$\forall xy(Rxy \supset Ryx)$
(w3) $\emptyset \vdash ((r \wedge (p \to q)) \to s) \vee ((p \wedge (r \to s)) \to q)$	$\forall xyz((Rxy \wedge Rxz) \supset (Ryz \vee Rzy))$

Theorem

Every weak strict implication logics generated by sequents in above Table is strongly complete with respect to its frames.

♠2. Bounded distributive lattices with implication

Definition

An algebra $\mathfrak{A} = (A, \wedge, \vee, \bot, \top, \to)$ is called a *bounded distributive lattice with implication* (BDI) if its $(\wedge, \vee, \bot, \top)$-reduct is a bounded distributive lattice and $\to$ is a binary operation on A satisfying the following conditions for all $a, b, c \in A$:

(C1) $(a \to b) \wedge (a \to c) = a \to (b \wedge c)$,

(C2) $(a \to c) \wedge (b \to c) = (a \vee b) \to c$.

(C3) $a \to \top = \top = \bot \to a$.

Bounded distributive lattices with implication

Definition (Celani & Jansana 2005)

A BDI $(A, \wedge, \vee, \bot, \top, \rightarrow)$ is called a *weak Heyting algebra* (WHA) if the following conditions are satisfied for all $a, b, c \in A$:

(C3) $\top = a \rightarrow a$

(C4) $(a \rightarrow b) \wedge (b \rightarrow c) \leq (a \rightarrow c)$

Let WH be the class of all WHAs.

Algebraic sequent system using simple sequents

Definition

The algebraic sequent system S_{BDI} consists of the following axiom schemata and rules:

$$\phi \vdash \phi, \quad \phi \vdash \top, \quad \bot \vdash \phi, \quad \top \vdash \alpha \to \top, \quad \top \vdash \bot \to \alpha$$

$$\text{(D)}\ \phi \wedge (\psi \vee \gamma) \vdash (\phi \wedge \psi) \vee (\phi \wedge \gamma),$$

$$\text{(M1)}\ (\phi \to \psi) \wedge (\phi \to \gamma) \vdash (\phi \to \gamma), \quad \text{(M2)}\ (\phi \to \gamma) \wedge (\psi \to \gamma) \vdash (\phi \vee \psi) \to \gamma,$$

$$\text{(M3)}\ \frac{\phi \vdash \psi}{\gamma \to \phi \vdash \gamma \to \psi}, \quad \text{(M4)}\ \frac{\phi \vdash \psi}{\psi \to \gamma \vdash \phi \to \gamma}, \quad \text{(cut)}\ \frac{\phi \vdash \psi \quad \psi \vdash \gamma}{\phi \vdash \gamma},$$

$$(\wedge\text{L})\ \frac{\phi_i \vdash \psi}{\phi_1 \wedge \phi_2 \vdash \psi}, \ (\wedge\text{R})\ \frac{\gamma \vdash \phi \quad \gamma \vdash \psi}{\gamma \vdash \phi \wedge \psi}, \ (\vee\text{L})\ \frac{\phi \vdash \gamma \quad \psi \vdash \gamma}{\phi \vee \psi \vdash \gamma}, \ (\vee\text{R})\ \frac{\psi \vdash \phi_i}{\psi \vdash \phi_1 \vee \phi_2}.$$

The i in $(\wedge\text{L})$ is equal to 1 or 2.

Weak Heyting Algebras

The algebraic sequent system $\mathsf{S}_{\mathsf{WH}} = \mathsf{S}_{\mathsf{BDI}} +$

$$\text{(I) } \psi \vdash \phi \to \phi, \quad \text{(Tr) } (\phi \to \psi) \wedge (\psi \to \gamma) \vdash \phi \to \gamma.$$

Theorem (Completeness)

For any $\phi \vdash \psi \in \mathcal{L}$ *and* $\mathcal{K} \in \{\mathsf{BDI}, \mathsf{WH}\}$, $\phi \vdash_{\mathsf{S}_{\mathcal{K}}} \psi$ *iff* $\mathcal{K} \models \phi \vdash \psi$.

Theorem

For every sequent $\Gamma \vdash \phi \in \mathcal{L}_S$, $\Gamma \vdash_{\mathsf{wK}_\sigma} \phi$ *iff* $\bigwedge \Gamma \vdash_{\mathsf{S}_{\mathsf{WH}}} \phi$.

Canonical extension of BDI

A *canonical extension* of a lattice L is a dense and compact completion of L. [Gehrke and Harding 2001].

Definition

Let $f : L \to M$ be any map from a lattice L to M. Define its canonical π-extension $f^{\pi} : L^{\delta} \to M^{\delta}$ by setting:

$$f^{\pi}(u) = \bigwedge\{\bigvee\{f(a) : a \in L \,\&\, x \leq a \leq y\} : K(L^{\delta}) \ni x \leq u \leq y \in O(L^{\delta})\}.$$

where $K(L^{\delta})$ and $O(L^{\delta})$ are sets of closed and open elements.

Proposition (Gehrke and Harding 2001)

Let $f : L \to M$ be an order-preserving map from a lattice L to M. Then f^{π} is order-preserving, and for all $u \in L^{\delta}$ and $y \in O(L^{\delta})$,

$$f^{\pi}(y) = \bigvee\{f(a) : L \ni a \leq y\}, \quad f^{\pi}(u) = \bigwedge\{f^{\pi}(y) : u \leq y \in O(L^{\delta})\}.$$

Canonical extension

The canonical extension of a BDI $(A, \to)$ is $(A^\delta, \to^\pi)$. We say that a class of algebras is *canonical* if it is closed under taking canonical extensions.

Theorem

BDI *is canonical.*

♠3. Algorithmic correspondence theory

Some references:

1. W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for distributive modal logic. *Annals of Pure and Applied Logic*, 163(3): 338-376, 2012.
2. W. Conradie, S. Ghilardi and A. Palmigiano. Unified correspondence. In A. Baltag and S. Smets (eds.) *Johan van Benthem on Logic and Information Dynamics*, pages, 933-976, Springer, 2014.

Expanded languages

$$\mathcal{L}_{term} \ni \phi ::= \top \mid \bot \mid p \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)$$
$$\mathcal{L}^{+}_{term} \ni \phi ::= \top \mid \bot \mid p \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid (\phi \cdot \phi)$$
$$\mathcal{L}^{*}_{term} \ni \phi ::= \top \mid \bot \mid p \mid \mathbf{i} \mid \mathbf{m} \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid (\phi \cdot \phi)$$

where $p \in$ Prop, $\mathbf{i} \in$ NOM and $\mathbf{m} \in$ CONOM. Nominals range over completely join-prime elements. Co-nominals range over completely meet-prime elements of a canonical extension.

Terms	inequalities (sequents)	quasi-inequalities (sequent rules)
$\mathcal{L}_{term}$	$\mathcal{L}$	$\mathcal{L}_{quasi}$
$\mathcal{L}^{+}_{term}$	$\mathcal{L}^{+}$	$\mathcal{L}^{+}_{quasi}$
$\mathcal{L}^{*}_{term}$	$\mathcal{L}^{*}$	$\mathcal{L}^{*}_{quasi}$

Expanded languages

Definition

Given a BDI $(A, \to)$, its canonical is $(A^\delta, \to^\pi)$, define

$$u \cdot^\delta v = \bigwedge \{w \in A^\delta \mid v \leq u \to^\pi w\}.$$

One can also define $u \leftarrow^\delta v = \bigvee \{w \in A^\delta \mid w \cdot^\delta v \leq u\}$.

Fact (Residuation)

$u \cdot^\delta v \leq w$ *iff* $v \leq u \to^\pi w$ *iff* $u \leq w \leftarrow^\pi v$

Inductive inequalities

Inductive inequalities are defined as standard. We need the classification of all nodes in a signed generation tree.

Table: Classification of nodes

Choice		Universal	
$+$	$\vee, \rightarrow$	$+$	$\rightarrow$
$-$	$\wedge, \rightarrow$		

Inductive inequalities

Definition

Given an order type ϵ and an irreflexive and transitive order Ω on the variable $p_1, \ldots, p_n$, the (negative or positive) generation tree $*\phi$ $(* \in \{+, 1\})$ of a formula $\phi(p_1, \ldots, p_n)$ is (Ω, ϵ)-inductive if, on every ϵ-critical branch with leaf p_i for $1 \leq i \leq n$, every choice node with a universal node as ancestor is binary, and hence labelled with $*(\phi \circ \psi)$, and

(i) $\epsilon^{\partial}(*\phi)$;

(ii) $p_j <_{\Omega} p_i$ for every p_j occurring in ϕ.

An inequality $\phi \leq \psi$ is (Ω, ϵ)-*inductive* if the trees $+\phi$ and $-\psi$ are both (Ω, ϵ)-inductive. An inequality $\phi \leq \psi$ is *inductive* if it is (Ω, ϵ)-inductive for some Ω and ϵ. When an inequality $\phi \leq \psi$ is inductive, we also say that the corresponding sequent $\phi \vdash \psi$ is inductive.

The algorithm StrictALBA

Stage 1. *Preprocessing & first approximation*

↓

Stage 2. *Reduction Elimination Cycle*

↓

Stage 3. *Output* (*pure quasi-inequality*)

The algorithm StrictALBA

Stage 1. Main Rules

- Spliting rules:

$$\frac{\phi \leq \psi \wedge \gamma}{\phi \leq \psi \quad \phi \leq \gamma} (\wedge\mathrm{Sp}) \qquad \frac{\phi \vee \psi \leq \gamma}{\phi \leq \gamma \quad \psi \leq \gamma} (\vee\mathrm{Sp})$$

- Approximation

$$\frac{\phi_i \leq \psi_i}{i_0 \leq \phi_i \quad \psi_i \leq m_0} (\mathrm{Ap})$$

The algorithm StrictALBA

Stage 2. Rules (a) Residuation rules:

$$\frac{\phi \cdot \psi \leq \gamma}{\phi \leq \psi \rightarrow \gamma} \text{(Res)}$$

(b) Approximation rule:

$$\frac{\phi \rightarrow \psi \leq m}{i \leq \phi \quad i \rightarrow \psi \leq m} (\rightarrow \text{Ap1}) \qquad \frac{\phi \rightarrow \psi \leq m}{\psi \leq n \quad \phi \rightarrow n \leq m} (\rightarrow \text{Ap2})$$

$$\frac{\phi \cdot \psi \leq m}{i \leq \phi \quad i \cdot \psi \leq m} (\cdot \text{Ap1}) \qquad \frac{\phi \cdot \psi \leq m}{i \leq \psi \quad \phi \cdot i \leq m} (\cdot \text{Ap2})$$

The algorithm StrictALBA

(c) Ackermann rules:

- The right Ackermann rule (RAck):

$$\left\{\begin{array}{l}\phi_1 \leq p \\ \vdots \\ \phi_n \leq p \\ \psi_1 \leq \gamma_1 \\ \vdots \\ \psi_m \leq \gamma_m\end{array}\right. \quad \text{is replaced with} \quad \left\{\begin{array}{l}\psi_1(\bigvee_{i=1}^n \phi_i/p) \leq \gamma_1(\bigvee_{i=1}^n \phi_i/p) \\ \vdots \\ \psi_m(\bigvee_{i=1}^n \phi_i/p) \leq \gamma_m(\bigvee_{i=1}^n \phi_i/p)\end{array}\right.$$

where (i) p does not occur in ϕ_i for $1 \leq i \leq n$; (ii) $\psi_j \leq \gamma_j$ is negative in p for $1 \leq j \leq m$.

The algorithm StrictALBA

- The left Ackermann rule (LAck):

$$\left\{\begin{array}{l} p \leq \phi_1 \\ \vdots \\ p \leq \phi_n \\ \psi_1 \leq \gamma_1 \\ \vdots \\ \psi_m \leq \gamma_m \end{array}\right. \quad \text{is replaced with} \quad \left\{\begin{array}{l} \psi_1(\bigwedge_{i=1}^n \phi_i/p) \leq \gamma_1(\bigwedge_{i=1}^n \phi_i/p) \\ \vdots \\ \psi_m(\bigwedge_{i=1}^n \phi_i/p) \leq \gamma_m(\bigwedge_{i=1}^n \phi_i/p) \end{array}\right.$$

where (i) p does not occur in ϕ_i for $1 \leq i \leq n$; (ii) each $\psi_j \leq \gamma_j$ positive in p for $1 \leq j \leq m$.

Example of StrictALBA

$(p \to q) \wedge (q \to r) \leq p \to r$. StrictALBA proceeds as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(p \to q) \wedge (q \to r) \leq p \to r}{\forall i \forall m (i \leq (p \to q) \wedge (q \to r) \;\&\; p \to r \leq m \Rightarrow i \leq m)}\,(\mathrm{Ap})}{\forall i \forall m (i \leq p \to q \;\&\; i \leq q \to r \;\&\; p \to r \leq m \Rightarrow i \leq m)}\,(\wedge\mathrm{Sp})}{\forall i \forall m (p \cdot i \leq q \;\&\; q \cdot i \leq r \;\&\; p \to r \leq m \Rightarrow i \leq m)}\,(\mathrm{Res})}{\forall ij \forall m (j \leq p \;\&\; j \cdot i \leq q \;\&\; q \cdot i \leq r \;\&\; p \to r \leq m \Rightarrow i \leq m)}\,(\cdot\mathrm{Ap1})}{\forall ij \forall m (j \cdot i \leq q \;\&\; q \cdot i \leq r \;\&\; j \to r \leq m \Rightarrow i \leq m)}\,(\mathrm{RAck})}{\forall ij \forall m ((j \cdot i) \cdot i \leq r \;\&\; j \to r \leq m \Rightarrow i \leq m)}\,(\mathrm{RAck})}{\forall ij \forall m (j \to ((j \cdot i) \cdot i) \leq m \Rightarrow i \leq m)}\,(\mathrm{RAck})
$$

First-order correspondent

Definition

(1) $\mathcal{M}, w \models \mathbf{i}$ iff $V(\mathbf{i}) = \{w\}$.

(2) $\mathcal{M}, w \models \mathbf{m}$ iff $V(\mathbf{m}) = W - \{w\}$.

(3) $\mathcal{M}, w \models p$ iff $w \in V(p)$.

(4) $\mathcal{M}, w \not\models \bot$.

(5) $\mathcal{M}, w \models \phi \wedge \psi$ iff $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$.

(6) $\mathcal{M}, w \models \phi \vee \psi$ iff $\mathcal{M}, w \models \phi$ or $\mathcal{M}, w \models \psi$.

(7) $\mathcal{M}, w \models \phi \rightarrow \psi$ iff $\forall u \in W(wRu \ \& \ \mathcal{M}, u \models \phi \Rightarrow \mathcal{M}, u \models \psi)$.

(8) $\mathcal{M}, w \models \phi \cdot \psi$ iff $\exists u \in W(uRw \ \& \ \mathcal{M}, w \models \phi \ \& \ \mathcal{M}, u \models \psi)$.

First-order correspondents

Given a frame $\mathcal{F} = (W, R)$, define a binary operator $\cdot$ on $\mathcal{P}(W)$ by setting

$$X \cdot Y = \{w \in W \mid \exists u(uRw \ \&\ w \in X \ \&\ u \in Y)\}$$

Then we have $[\![\phi \cdot \psi]\!]_{\mathcal{M}} = [\![\phi]\!]_{\mathcal{M}} \cdot [\![\psi]\!]_{\mathcal{M}}$.
Moreover, we have the following fact:

Proposition

For any $X, Y, Z \in \mathcal{P}(W)$, $X \cdot Y \subseteq Z$ iff $Y \subseteq X \to Z$.

First-order correspondents

Obviously, the output pure quasi-inequality $\forall ij \forall m(j \rightarrow ((j \cdot i) \cdot i) \leq m \Rightarrow i \leq m)$ is equivalent to $\forall ij(j \cdot i \leq (j \cdot i) \cdot i)$. Notice that $z \in \{x\} \cdot \{y\}$ iff Ryx. Then the first-order condition is calculated as follows:

$$
\begin{aligned}
\forall ij(j \cdot i \leq (j \cdot i) \cdot i) &\Leftrightarrow \forall xy(\{x\} \cdot \{y\} \subseteq (\{x\} \cdot \{y\}) \cdot \{y\}) \\
&\Leftrightarrow \forall xyz(z \in \{x\} \cdot \{y\} \supset z \in (\{x\} \cdot \{y\}) \cdot \{y\}) \\
&\Leftrightarrow \forall xyz(Ryx \supset \exists u(Ruz \wedge z \in (\{x\} \cdot \{y\}) \wedge u \in \{y\}) \\
&\Leftrightarrow \forall xyz(Ryx \supset (Ryz \wedge z \in (\{x\} \cdot \{y\})) \\
&\Leftrightarrow \forall xyz(Ryx \supset (Ryx \wedge Ryx))
\end{aligned}
$$

which is a tautology. The sequent $p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$ is an axiom of wK_σ.

First-order correspondents

Example: $p, p \rightarrow q \vdash q$. By StrictALBA one gets

$$\forall i(i \leq i \cdot i)$$

The first-order condition is calculated as follows:

$$\begin{aligned}
\forall i(i \leq i \cdot i) &\Leftrightarrow \forall x(\{x\} \subseteq \{x\} \cdot \{x\}) \\
&\Leftrightarrow \forall xz(z \in \{x\} \supset z \in \{x\} \cdot \{x\}) \\
&\Leftrightarrow \forall xRxx
\end{aligned}$$

It follows that the sequent $p \wedge (p \rightarrow q) \vdash q$ defines the class of all reflexive frames.

Canonicity of Inductive Inequalities

Theorem

All inductive $\mathcal{L}_{term}$-inequalities are canonical.

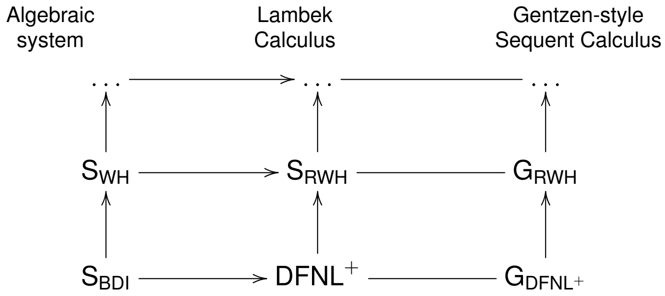
Proof.

We can use the U-shaped argument represented below to show that from $\mathfrak{A} \models \phi \leq \psi$ we can get $\mathfrak{A}^\delta \vdash \phi \leq \psi$:

$$\begin{array}{ccc} \mathfrak{A} \models_{\mathfrak{A}} \phi \leq \psi & & \mathfrak{A}^\delta \vdash \phi \leq \psi \\ \Updownarrow & & \Updownarrow \\ \mathfrak{A}^\delta \vdash_{\mathfrak{A}} \text{ALBA}(\phi \leq \psi) & \Leftrightarrow & \mathfrak{A}^\delta \vdash \text{ALBA}(\phi \leq \psi). \end{array}$$

See [Conradie and Palmigiano 2012]. □

♠4. Conservativity



The Lambek calculi we considered are non-associative extensions of $DFNL^{+}$.

Lattice-ordered residuated groupoid

Definition

A *bounded distributive lattice-ordered residuated groupoid* (BDRG) is an algebra $\mathfrak{A} = (A, \wedge, \vee, \top, \bot, \rightarrow, \cdot, \leftarrow)$ where $(A, \wedge, \vee, \top, \bot)$ is a bounded distributive lattice, and $\cdot, \rightarrow, \leftarrow$ are binary operations on A satisfying the following residuation law for all $a, b, c \in A$:

$$\text{(RES) } a \cdot b \leq c \text{ iff } b \leq a \rightarrow c \text{ iff } a \leq c \leftarrow b.$$

Let BDRG be the class of all BDRGs.

DFNL$^+$

Definition (Buskowski 2006)

An algebraic sequent calculus DFNL$^+$ for BDRG consists of the following axiom schemata and rules:

$$\text{(Id) } \phi \vdash \phi, \quad (\top)\ \phi \vdash \top, \quad (\bot)\ \bot \vdash \phi, \quad \text{(D) } \phi \wedge (\psi \vee \gamma) \vdash (\phi \wedge \psi) \vee (\phi \wedge \gamma),$$

$$(\wedge\text{L}) \ \frac{\phi_i \vdash \psi}{\phi_1 \wedge \phi_2 \vdash \psi}\ (i = 1, 2), \quad (\wedge\text{R}) \ \frac{\gamma \vdash \phi \quad \gamma \vdash \psi}{\gamma \vdash \phi \wedge \psi},$$

$$(\vee\text{L}) \ \frac{\phi \vdash \gamma \quad \psi \vdash \gamma}{\phi \vee \psi \vdash \gamma}, \quad (\vee\text{R}) \ \frac{\psi \vdash \phi_i}{\psi \vdash \phi_1 \vee \phi_2}\ (i = 1, 2), \quad \text{(cut) } \frac{\phi \vdash \psi \quad \psi \vdash \gamma}{\phi \vdash \gamma},$$

$$\text{(Res1) } \frac{\phi \cdot \psi \vdash \gamma}{\psi \vdash \phi \to \psi}, \ \text{(Res2) } \frac{\psi \vdash \phi \to \psi}{\phi \cdot \psi \vdash \gamma}, \ \text{(Res3) } \frac{\phi \cdot \psi \vdash \gamma}{\phi \vdash \gamma \leftarrow \psi}, \ \text{(Res4) } \frac{\phi \vdash \gamma \leftarrow \psi}{\phi \cdot \psi \vdash \gamma}.$$

Consequence relation for DFNL$^+$

Definition

An $\mathcal{L}^+$-*supersequent* is an expression of the form $\Phi \Rightarrow \chi \vdash \delta$ (consequence relation) where $\Phi \cup \{\chi \vdash \delta\} \subseteq \mathcal{L}^+$.

Theorem (Strong completeness)

For every $\mathcal{L}^+$-supersequent sequents $\Phi \Rightarrow \chi \vdash \psi$, $\vdash_{\mathrm{DFNL}^+} \Phi \Rightarrow \chi \vdash \delta$ iff BDRG $\models \Phi \Rightarrow \chi \vdash \delta$.

Conservativity: from $\mathsf{S}_{\mathsf{BDI}}$ to DFNL^+

Lemma

For every BDRG $(A, \rightarrow, \cdot, \leftarrow)$, its $(\wedge, \vee, \bot, \top, \rightarrow)$-reduct is a BDI.

Lemma

For any BDI $(A, \rightarrow)$, the algebra $(A^\delta, \rightarrow^\pi, \cdot^\delta, \leftarrow^\pi)$ is a BDRG.

Theorem (conservativity)

For every sequent $\phi \vdash \psi \in \mathcal{L}$, $\phi \vdash_{\mathsf{S}_{\mathsf{BDI}}} \psi$ iff $\phi \vdash_{\mathsf{DFNL}^+} \psi$.

Proof.

Let $\phi \vdash \psi \in \mathcal{L}$. Obviously, $\phi \vdash_{\mathsf{S}_{\mathsf{BDI}}} \psi$ implies $\phi \vdash_{\mathsf{DFNL}^+} \psi$. Conversely, assume $\phi \nvdash_{\mathsf{S}_{\mathsf{BDI}}} \psi$. By the completeness of $\mathsf{S}_{\mathsf{BDI}}$, there exist an BDI $\mathfrak{A} = (A, \rightarrow)$ and an assignment μ such that $\mu(\phi) \nleq \mu(\psi)$. Consider $\mathfrak{A}^\delta = (A^\delta, \rightarrow^\pi, \cdot^\delta, \leftarrow^\delta)$. Then $\mathfrak{A}^\delta, \mu \not\models \phi \vdash \psi$. Then BDRG $\not\models \phi \vdash \psi$. By the completeness of DFNL^+, $\phi \nvdash_{\mathsf{DFNL}^+} \psi$. □

Conservativity

Residuated weak Heyting algebra: BDRG satisfying

$$(w)\ a \cdot b \leq a, \quad (ct)\ a \cdot b \leq (a \cdot b) \cdot b$$

$\mathsf{S}_{\mathsf{RWH}} = \mathsf{DFNL}^{+} +$

$$\phi \cdot \psi \vdash \phi, \quad \phi \cdot \psi \leq (\phi \cdot \psi) \cdot \psi$$

Lemma

For every RWH-algebra $\mathfrak{A} = (A, \wedge, \vee, \bot, \top, \rightarrow, \cdot, \leftarrow)$, its $(\wedge, \vee, \bot, \top, \rightarrow)$-reduct is a WH-algebra.

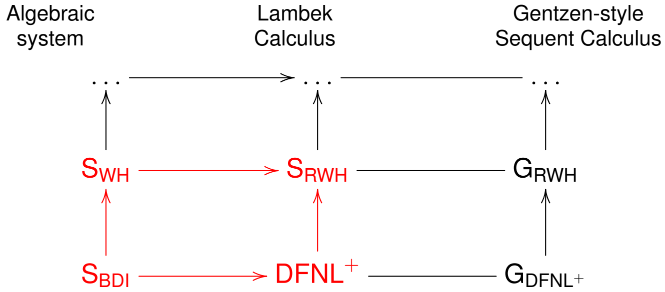
Lemma

For any WH-algebra $(A, \rightarrow)$, $(A^{\delta}, \rightarrow^{\pi}, \cdot^{\delta}, \leftarrow^{\pi})$ is a RWH-algebra.

Theorem (conservativity)

For every sequent $\phi \vdash \psi \in \mathcal{L}$, $\phi \vdash_{\mathsf{S}_{\mathsf{WH}}} \psi$ iff $\phi \vdash_{\mathsf{RWH}} \psi$.

Conservativity



Ackermann Lemma Based Calculus

Extensions of $\mathrm{S_{BDI}}$ and $\mathrm{DFNL^+}$:

Example

1. $(Tr)\ (p \to q) \wedge (q \to r) \vdash p \to r$ corresponds to $(Tr')\ p \cdot q \vdash (p \cdot q) \cdot q$, i.e., they define the same class of BDRGs.
2. $(W)\ q \vdash p \to p$ corresponds to $(W')\ p \cdot q \vdash p$.
3. $\mathrm{S_{WH}} = \mathrm{S_{BDI}} + (W) + (Tr)$ is conservatively extended to $\mathrm{DFNL^+} + (Tr') + (W')$

An Ackermann Lemma Based Calculus to calculate the algebraic correspondence between sequents.

Ackermann Lemma Based Calculus

Definition

The Ackermann lemma based supersequent calculus ALC:

(1) Splitting rules:

$$(\wedge S)\ \frac{\gamma \vdash \phi, \gamma \vdash \psi, \Gamma \Rightarrow \chi \vdash \delta}{\gamma \vdash \phi \wedge \psi, \Gamma \Rightarrow \chi \vdash \delta} \qquad (\wedge S\!\uparrow)\ \frac{\gamma \vdash \phi \wedge \psi, \Gamma \Rightarrow \chi \vdash \delta}{\gamma \vdash \phi, \gamma \vdash \psi, \Gamma \Rightarrow \chi \vdash \delta}$$

$$(\vee S)\ \frac{\phi \vdash \gamma, \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\phi \vee \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta} \qquad (\vee S\!\uparrow)\ \frac{\phi \vee \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\phi \vdash \gamma, \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}$$

Ackermann Lemma Based Calculus

(2) Residuation rules:

$$(\mathrm{ReL1})\ \frac{\phi \cdot \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\psi \vdash \phi \to \gamma, \Gamma \Rightarrow \chi \vdash \delta} \qquad (\mathrm{ReL1}\!\uparrow)\ \frac{\psi \vdash \phi \to \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\phi \cdot \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}$$

$$(\mathrm{ReL2})\ \frac{\phi \cdot \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\phi \vdash \gamma \leftarrow \psi, \Gamma \Rightarrow \chi \vdash \delta} \qquad (\mathrm{ReL2}\!\uparrow)\ \frac{\phi \vdash \gamma \leftarrow \psi, \Gamma \Rightarrow \chi \vdash \delta}{\phi \cdot \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}$$

$$(\mathrm{ReR1})\ \frac{\Gamma \Rightarrow \phi \cdot \psi \vdash \gamma}{\Gamma \Rightarrow \psi \vdash \phi \to \gamma} \qquad (\mathrm{ReR1}\!\uparrow)\ \frac{\Gamma \Rightarrow \psi \vdash \phi \to \gamma}{\Gamma \Rightarrow \phi \cdot \psi \vdash \gamma}$$

$$(\mathrm{ReR2})\ \frac{\Gamma \Rightarrow \phi \cdot \psi \vdash \gamma}{\Gamma \Rightarrow \phi \vdash \gamma \leftarrow \psi} \qquad (\mathrm{ReR2}\!\uparrow)\ \frac{\Gamma \Rightarrow \phi \vdash \gamma \leftarrow \psi}{\Gamma \Rightarrow \phi \cdot \psi \vdash \gamma}$$

Ackermann Lemma Based Calculus

(3) Approximation rules:

$$(\mathrm{AAp1})\ \frac{p \vdash \phi, \Gamma \Rightarrow p \vdash \psi}{\Gamma \Rightarrow \phi \vdash \psi} \qquad (\mathrm{AAp1}\uparrow)\ \frac{\Gamma \Rightarrow \phi \vdash \psi}{p \vdash \phi, \Gamma \Rightarrow p \vdash \psi}$$

$$(\mathrm{AAp2})\ \frac{\psi \vdash p, \Gamma \Rightarrow \phi \vdash p}{\Gamma \Rightarrow \phi \vdash \psi} \qquad (\mathrm{AAp2}\uparrow)\ \frac{\Gamma \Rightarrow \phi \vdash \psi}{\psi \vdash p, \Gamma \Rightarrow \phi \vdash p}$$

$$(\to \mathrm{Ap1})\ \frac{p \vdash \phi, p \to \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\phi \to \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta} \qquad (\to \mathrm{Ap1}\uparrow)\ \frac{\phi \to \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}{p \vdash \phi, p \to \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}$$

$$(\to \mathrm{Ap2})\ \frac{\psi \vdash p, \phi \to p \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\phi \to \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta} \qquad (\to \mathrm{Ap2}\uparrow)\ \frac{\phi \to \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\psi \vdash p, \phi \to p \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}$$

$$(\to \mathrm{Ap3})\ \frac{\phi \vdash p, \gamma \vdash p \to \psi, \Gamma \Rightarrow \chi \vdash \delta}{\gamma \vdash \phi \to \psi, \Gamma \Rightarrow \chi \vdash \delta} \qquad (\to \mathrm{Ap3}\uparrow)\ \frac{\gamma \vdash \phi \to \psi, \Gamma \Rightarrow \chi \vdash \delta}{\phi \vdash p, \gamma \vdash p \to \psi, \Gamma \Rightarrow \chi \vdash \delta}$$

$$(\to \mathrm{Ap4})\ \frac{p \vdash \psi, \gamma \vdash \phi \to p, \Gamma \Rightarrow \chi \vdash \delta}{\gamma \vdash \phi \to \psi, \Gamma \Rightarrow \chi \vdash \delta} \qquad (\to \mathrm{Ap4}\uparrow)\ \frac{\gamma \vdash \phi \to \psi, \Gamma \Rightarrow \chi \vdash \delta}{p \vdash \psi, \gamma \vdash \phi \to p, \Gamma \Rightarrow \chi \vdash \delta}$$

Ackermann Lemma Based Calculus

$$(\cdot\text{Ap1})\ \frac{p \vdash \psi, \phi \vdash p \cdot \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\phi \vdash \psi \cdot \gamma, \Gamma \Rightarrow \chi \vdash \delta} \qquad (\cdot\text{Ap1}\uparrow)\ \frac{\phi \vdash \psi \cdot \gamma, \Gamma \Rightarrow \chi \vdash \delta}{p \vdash \psi, \phi \vdash p \cdot \gamma, \Gamma \Rightarrow \chi \vdash \delta}$$

$$(\cdot\text{Ap2})\ \frac{p \vdash \gamma, \phi \vdash \psi \cdot p, \Gamma \Rightarrow \chi \vdash \delta}{\phi \vdash \psi \cdot \gamma, \Gamma \Rightarrow \chi \vdash \delta} \qquad (\cdot\text{Ap2}\uparrow)\ \frac{\phi \vdash \psi \cdot \gamma, \Gamma \Rightarrow \chi \vdash \delta}{p \vdash \gamma, \phi \vdash \psi \cdot p, \Gamma \Rightarrow \chi \vdash \delta}$$

$$(\cdot\text{Ap3})\ \frac{\phi \vdash p, p \cdot \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\phi \cdot \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta} \qquad (\cdot\text{Ap3}\uparrow)\ \frac{\phi \cdot \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\phi \vdash p, p \cdot \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}$$

$$(\cdot\text{Ap4})\ \frac{\psi \vdash p, \phi \cdot p \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\phi \cdot \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta} \qquad (\cdot\text{Ap4}\uparrow)\ \frac{\phi \cdot \psi \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}{\psi \vdash p, \phi \cdot p \vdash \gamma, \Gamma \Rightarrow \chi \vdash \delta}$$

where p, q do not occur in the conclusion.

Ackermann Lemma Based Calculus

(4) Ackermann rules:

$$(\text{RAck})\ \frac{\Gamma[\bigvee_{i=1}^{n} \phi_i/p], \Gamma' \Rightarrow (\chi \vdash \delta)^*}{\phi_1 \vdash p, \ldots, \phi_n \vdash p, \Gamma, \Gamma' \Rightarrow \chi \vdash \delta}$$

$$(\text{RAck}\uparrow)\ \frac{\phi_1 \vdash p, \ldots, \phi_n \vdash p, \Gamma, \Gamma' \Rightarrow \chi \vdash \delta}{\Gamma[\bigvee_{i=1}^{n} \phi_i/p], \Gamma' \Rightarrow (\chi \vdash \delta)^*}$$

where (i) p does not occur in Γ' or ϕ_i for $1 \leq i \leq n$; (ii) $\Gamma = \{\psi_j \vdash \gamma_j \mid \psi_j(+p), \gamma_j(-p), 1 \leq j \leq m\}$ and

$$\Gamma[\bigvee_{i=1}^{n} \phi_i/p] = \{\psi_j[\bigvee_{i=1}^{n} \phi_i/p] \vdash \gamma_j[\bigvee_{i=1}^{n} \phi_i/p] \mid \psi_j \vdash \gamma_j \in \Gamma\}$$

and (iii) either p does not occur in $\chi \vdash \delta$ and $(\chi \vdash \delta)^* = x \vdash \delta$, or $\chi \vdash \delta$ is negative in p and $(\chi \vdash \delta)^* = \chi[\bigvee_{i=1}^{n} \phi_i/p] \vdash \delta[\bigvee_{i=1}^{n} \phi_i/p]$.

Ackermann Lemma Based Calculus

$$\text{(LAck)}\ \frac{\Gamma[\bigwedge_{i=1}^{n}\phi_i/p], \Gamma' \Rightarrow (\chi \vdash \delta)^*}{p \vdash \phi_1, \ldots, p \vdash \phi_n, \Gamma, \Gamma' \Rightarrow \chi \vdash \delta}$$

$$\text{(LAck}\uparrow\text{)}\ \frac{p \vdash \phi_1, \ldots, p \vdash \phi_n, \Gamma, \Gamma' \Rightarrow \chi \vdash \delta}{\Gamma[\bigwedge_{i=1}^{n}\phi_i/p], \Gamma' \Rightarrow (\chi \vdash \delta)^*}$$

where (i) p does not occur in Γ' or ϕ_i for $1 \leq i \leq n$; (ii) $\Gamma = \{\psi_j \vdash \gamma_j \mid \psi_j(-p), \gamma_j(+p), 1 \leq j \leq m\}$ and

$$\Gamma[\bigwedge_{i=1}^{n}\phi_i/p] = \{\psi_j[\bigwedge_{i=1}^{n}\phi_i/p] \vdash \gamma_j[\bigwedge_{i=1}^{n}\phi_i/p] \mid \psi_j \vdash \gamma_j \in \Gamma\}$$

and (iii) either p does not occur in $\chi \vdash \delta$ and $(\chi \vdash \delta)^* = x \vdash \delta$, or $\chi \vdash \delta$ is positive in p and $(\chi \vdash \delta)^* = \chi[\bigwedge_{i=1}^{n}\phi_i/p] \vdash \delta[\bigwedge_{i=1}^{n}\phi_i/p]$.

Algebraic correspondence

Algebraic correspondence between $\mathcal{L}$ and sequents in $\mathcal{L}^\bullet$:

$$\mathcal{L}^\bullet_{term} \ni \phi ::= p \mid \top \mid \bot \mid (\phi \cdot \phi)$$

Definition

Given sequents $\phi \vdash \psi \in \mathcal{L}$ and $\chi \vdash \delta \in \mathcal{L}^\bullet$, we say that $\phi \vdash \psi$ *corresponds* to $\chi \vdash \delta$ over BDRG if they define the same class of BDRGs.

Fact

Given sequents $\phi \vdash \psi \in \mathcal{L}$ and $\chi \vdash \delta \in \mathcal{L}^\bullet$, if the following rule

$$\frac{\Rightarrow \phi \vdash \psi}{\Rightarrow \chi \vdash \delta}(r)$$

is derivable in ALC*, then $\phi \vdash \psi$ corresponds to $\chi \vdash \delta$.*

Examples

(Tr) One proof is as follows:

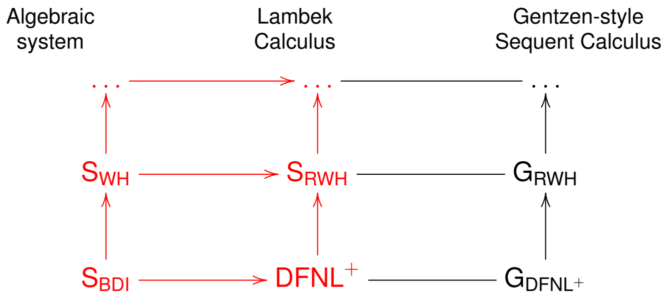
$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow (p \to q) \wedge (q \to r) \vdash (p \to r)}{s \vdash (p \to q) \wedge (q \to r) \Rightarrow s \vdash p \to r}\ (\text{AAp1}\uparrow)}{s \vdash p \to q, s \vdash q \to r \Rightarrow s \vdash p \to r}\ (\wedge\text{S}\uparrow)}{p \cdot s \vdash q, q \cdot s \vdash r \Rightarrow p \cdot s \vdash r}\ (\text{ReL1}\uparrow, \text{ReR1}\uparrow)}{p \cdot s \vdash q \Rightarrow p \cdot s \vdash q \cdot s}\ (\text{AAp2})}{\Rightarrow p \cdot s \vdash (p \cdot s) \cdot s}\ (\text{RAck}\uparrow)
$$

Conservativity

Theorem (Conservativity)

Assume that Φ *is a set of inductive sequents in* $\mathcal{L}$, *and* $\Psi \subseteq \mathcal{L}^{\bullet}$ *is the set of correspondents of sequents in* Φ. *Then the algebraic sequent system* $\mathsf{DFNL}^{+}(\Psi)$ *is a conservative extension of* $\mathsf{S}_{\mathsf{BDI}}(\Phi)$.

♠5. Gentzen-style sequent calculi



Cut-free Gentzen-style sequent calculus for DFNL$^+$

The Gentzen-style sequent calculus G_{DFNL^+}:

$$(\mathrm{Id})\ \phi \vdash \phi, \quad (\top)\ \Gamma \vdash \top, \quad (\bot)\ \Gamma[\bot] \vdash \phi,$$

$$(\to \mathrm{L})\ \dfrac{\Delta \vdash \phi \quad \Gamma[\psi] \vdash \gamma}{\Gamma[\Delta \odot (\phi \to \psi)] \vdash \gamma}, \qquad (\to \mathrm{R})\ \dfrac{\phi \odot \Gamma \vdash \psi}{\Gamma \vdash \phi \to \psi},$$

$$(\leftarrow \mathrm{L})\ \dfrac{\Gamma[\phi] \vdash \gamma \quad \Delta \vdash \psi}{\Gamma[(\phi \leftarrow \psi) \odot \Delta] \vdash \gamma}, \qquad (\leftarrow \mathrm{R})\ \dfrac{\Gamma \odot \psi \vdash \phi}{\Gamma \vdash \phi \leftarrow \psi},$$

$$(\cdot \mathrm{L})\ \dfrac{\Gamma[\phi \odot \psi] \vdash \gamma}{\Gamma[\phi \cdot \psi] \vdash \gamma}, \qquad (\cdot \mathrm{R})\ \dfrac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma \odot \Delta \vdash \phi \cdot \psi},$$

$$(\wedge \mathrm{L})\ \dfrac{\Gamma[\phi \owedge \psi] \vdash \gamma}{\Gamma[\phi \wedge \psi] \vdash \gamma}, \qquad (\wedge \mathrm{R})\ \dfrac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma \owedge \Delta \vdash \phi \wedge \psi},$$

$$(\vee \mathrm{L})\ \dfrac{\Gamma[\phi] \vdash \gamma, \quad \Gamma[\psi] \vdash \gamma}{\Gamma[\phi \vee \psi] \vdash \gamma}, \qquad (\vee \mathrm{R})\ \dfrac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2} (i = 1, 2),$$

$$(\owedge \mathrm{C})\ \dfrac{\Gamma[\Delta \owedge \Delta] \vdash \phi}{\Gamma[\Delta] \vdash \phi}, \qquad (\owedge \mathrm{W})\ \dfrac{\Gamma[\Delta] \vdash \phi}{\Gamma[\Sigma \owedge \Delta] \vdash \phi}, \qquad (\owedge \mathrm{E})\ \dfrac{\Gamma[\Delta \owedge \Lambda] \vdash \phi}{\Gamma[\Lambda \owedge \Delta] \vdash \phi},$$

$$(\owedge \mathrm{As})\ \dfrac{\Gamma[(\Delta_1 \owedge \Delta_2) \owedge \Delta_3] \vdash \phi}{\Gamma[\Delta_1 \owedge (\Delta_2 \owedge \Delta_3)] \vdash \phi}.$$

Cut-free sequent calculus for extensions of DFNL$^+$

Given $\chi \vdash \delta \in \mathcal{L}^\bullet$, define the rule

$$\frac{\delta[\Delta_1/p_1, \ldots, \Delta_n/p_n] \Rightarrow \phi}{\chi[\Delta_1/p_1, \ldots, \Delta_n/p_n] \Rightarrow \phi}(\odot\sigma)$$

where $\delta[\Delta_1/p_1, \ldots, \Delta_n/p_n$ and $\chi[\Delta_1/p_1, \ldots, \Delta_n/p_n]$ are obtained from δ and χ by substituting Δ_i for p_i, and $\odot$ for $\cdot$ uniformly.

Example

For $p \cdot q \vdash (p \cdot q) \cdot q$, we have the following rule:

$$\frac{(\Delta \odot \Sigma) \odot \Sigma \Rightarrow \phi}{\Delta \odot \Sigma \Rightarrow \phi}$$

Gentzen-style Sequent Calculi

For any set of sequents $\Psi \subseteq \mathcal{L}^\bullet$, let $\odot\Psi = \{\odot\sigma \mid \sigma \in \Psi\}$ and $\mathsf{G}_{\mathsf{DFNL}^+}(\odot\Psi)$ be the Gentzen-style sequent system obtained from $\mathsf{G}_{\mathsf{DFNL}^+}$ by adding rules in $(\odot\Psi)$.

Theorem

For any set of sequents $\Psi \subseteq \mathcal{L}^\bullet$, the (cut) rule is admissible in the Gentzen-style sequent system $\mathsf{G}_{\mathsf{DFNL}^+}(\odot\Psi)$.

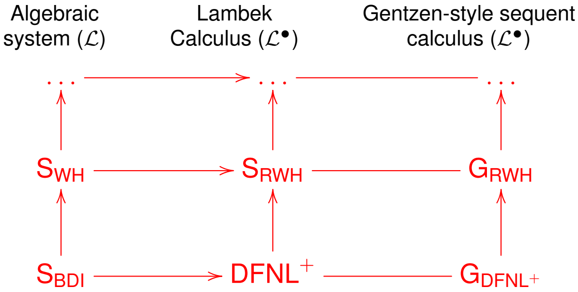
Theorem

For any set of sequents $\Psi \subseteq \mathcal{L}^\bullet$, the following hold:

(1) $\Gamma \vdash_{\mathsf{G}_{\mathsf{DFNL}^+}(\odot\Psi)} \phi$ *iff* $\mathsf{Alg}(\Psi) \models \Gamma \vdash \phi$.

(2) *if every subformula of δ is a subformula of χ for each sequent $\chi \vdash \delta \in \Psi$, then $\mathsf{G}_{\mathsf{DFNL}^+}(\odot\Psi)$ has the subformula property.*

Gentzen-style sequent calculi



Further Work

1. Extend the algebraic correspondence between $\mathcal{L}$ and $\mathcal{L}^{\bullet}$.
2. Logics weaker than S_{BDI} and Lambek Calculi below $DFNL^{+}$.
3. The relational semantics for S_{BDI}.
4. Duality theory that generalises [Celani & Jansana 2005]

Thanks for your attention!