TiRS graphs and frames: a new setting for duals of canonical extensions of lattices

Andrew Craig[1] Maria Gouveia [2] Miroslav Haviar[3]

[1]University of Johannesburg, Johannesburg, South Africa

[2]University of Lisbon, Lisbon, Portugal

[3]Matej Bel University, Banská Bystrica, Slovakia

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This can be achieved by using: (i) in distributive case: Priestley duality as a natural duality, and (ii) in non-distributive case: a topological representation of BLs due to M. Ploščica (1995) which presents **the classical one due to A. Urquhart (1978)** in the spirit of the natural dualities.

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Definition

Let $\mathbf{B}$ be a Boolean algebra (with operators) and let $X_{\mathbf{B}}$ be the Stone space dual to $\mathbf{B}$, i.e., $X_{\mathbf{B}}$ is the set of ultrafilters of $\mathbf{B}$ with an appropriate topology. (Stone duality tells us that we may identify the Boolean algebra $\mathbf{B}$ with the Boolean algebra of *clopen* subsets of the Stone space $X_{\mathbf{B}}$.)

The canonical extension $\mathbf{B}^{\delta}$ of $\mathbf{B}$ is the Boolean algebra $\wp(X_{\mathbf{B}})$ of *all* subsets of the set $X_{\mathbf{B}}$ of ultrafilters of $\mathbf{B}$ (with the operators extended in a natural way).

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Thus, roughly speaking, Jónsson and Tarski obtained $\mathbf{B}^{\delta}$ from the Stone space $X_{\mathbf{B}}$ by forgetting the topology.

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- When the members of the class of lattice-based algebras are the algebraic models of a logic, canonicity leads to completeness results for the associated logic.
- That is partly why the canonical extensions are important and have been of a great interest during the last two decades.

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The canonical extension $\mathbf{L}^{\delta}$ of $\mathbf{L}$ is the **doubly algebraic distributive** lattice $\mathrm{Up}(X_{\mathbf{L}})$ of *all* up-sets of the ordered set $\langle X_{\mathbf{L}}; \subseteq \rangle$ of prime filters of $\mathbf{L}$.

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Thus, again, $\mathbf{L}^{\delta}$ is obtained from the Priestley space $X_{\mathbf{L}}$ by forgetting the topology.

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- $\mathbf{C}$ is called a compact completion of $\mathbf{L}$ if, for every filter F of $\mathbf{L}$ and every ideal J of $\mathbf{L}$, we have $\bigwedge F \leqslant \bigvee J$ implies $F \cap J \neq \varnothing$.

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- Abstractly, **a canonical extension of a BL** $\mathbf{L}$ has been defined as a **dense** and **compact completion** of $\mathbf{L}$.
- Concretely, they constructed $\mathbf{L}^\delta$ as the complete lattice of **Galois-closed sets of the polarity** between the filter lattice $\mathrm{Filt}(\mathbf{L})$ and the ideal lattice $\mathrm{Idl}(\mathbf{L})$ of $\mathbf{L}$:

$$(F, I) \in R \iff F \cap I \neq \emptyset.$$

Ploščica's representation for bounded lattices

- Let $\mathbf{L}$ be a bounded lattice. Ploščica's dual of $\mathbf{L}$ is $D(\mathbf{L}) = \mathbb{X}_{\mathbf{L}} := (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R, \mathcal{T})$

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$$(f, g) \in R \quad \text{iff} \quad f^{-1}(1) \cap g^{-1}(0) = \emptyset.$$

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  Equivalently, $(f, g) \in R$ iff $\forall a \in dom f \cap dom\, g, f(a) \leqslant g(a)$. The topology $\mathcal{T}$ has as a subbasis of closed sets $\{ V_a, W_a \mid a \in L \}$, with $V_a = \{f \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 0 \}$ and $W_a = \{f \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 1 \}$.

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- Ploščica's second dual of $\mathbf{L}$ is $ED(\mathbf{L}) := \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbb{X}_{\mathbf{L}}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$, the set of all continuous maximal partial R-preserving maps from $\mathbb{X}_{\mathbf{L}} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R, \mathcal{T})$ to $\underset{\sim}{\mathbf{2}}_{\mathcal{T}} = (\{0, 1\}, \leqslant, \mathcal{T})$.

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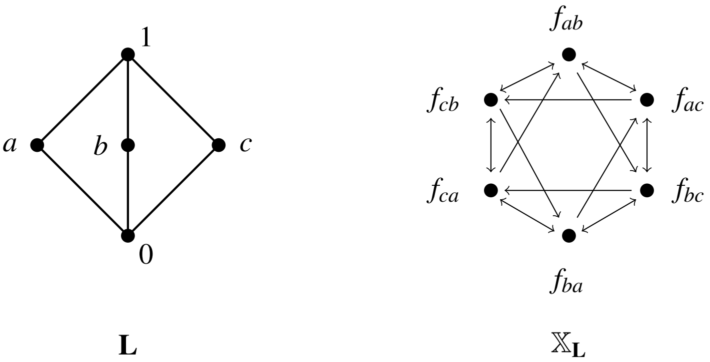
Theorem (Ploščica, 1995)

Let $\mathbf{L} \in \mathcal{L}$. Then $\mathbf{L} \cong ED(\mathbf{L})$ via the map $a \mapsto e_a$ where $e_a \colon (\mathbb{X}_{\mathbf{L}}, \mathcal{T}) \to \underset{\sim}{\mathbf{2}}_{\mathcal{T}}$ is defined by $e_a(f) = f(a)$.

Example of the dual graph of a bounded lattice



The modular lattice $\mathbf{L} = \mathbf{M}_3$ and its graph $\mathbb{X}_{\mathbf{L}} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R)$.

We define $f_{xy} \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ by $f_{xy}^{-1}(1) = {\uparrow}x$ and $f_{xy}^{-1}(0) = {\downarrow}y$.

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(S) for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$,
- (i) $x_1 \neq x_2$ implies $x_1R \neq x_2R$;
- (ii) $y_1 \neq y_2$ implies $Ry_1 \neq Ry_2$.

(R)
- (i) for every $x \in X$ there exists $y \in Y$ such that $\neg(xRy)$ and $\forall w \in X\ ((w \neq x \ \ \&\ \ xR \subseteq wR) \Rightarrow wRy)$;
- (ii) for every $y \in Y$ there exists $x \in X$ such that $\neg(xRy)$ and $\forall z \in Y\ ((z \neq y \ \ \&\ \ Ry \subseteq Rz) \Rightarrow xRz)$.

Duality: perfect lattices vs RS frames (Gehrke, 2006)

From perfect lattices to RS frames: Let $\mathbf{C}$ be a perfect lattice. Then the mapping

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From RS frames to perfect lattices: Let $\mathbb{F} = (X, Y, R)$ be an RS frame. A Galois connection between $\wp(X)$ and $\wp(Y)$ is defined as follows for $A \subseteq X$, $B \subseteq Y$:

$$R_{\triangleright}(A) = \{\, y \in Y \mid \forall\, a \in A, aRy \,\}\ R_{\triangleleft}(B) = \{\, x \in X \mid \forall b \in B, xRb \,\}.$$

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Then $\mathcal{G}(\mathbb{F}) := \{\, A \subseteq X \mid A = R_{\triangleleft} \circ R_{\triangleright}(A) \,\}$ is a perfect lattice.

Defining RS graphs (Craig, Gouveia, MH, 2015)

Lemma

Let $\mathbf{L}$ *be a bounded lattice and let* $\mathbb{X}_{\mathbf{L}} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R) = (X, R)$. *Then* $\mathbb{X}$ *satisfies the conditions below:*

(S) *for every* $f, g \in X$, *if* $f \neq g$ *then* $f^{-1}(1) \neq g^{-1}(1)$ *or* $f^{-1}(0) \neq g^{-1}(0)$;

(R) (i) *for all* $f, h \in X$, *if* $f^{-1}(1) \subsetneq h^{-1}(1)$ *then* $h^{-1}(1) \cap f^{-1}(0) \neq \emptyset$;
(ii) *for all* $g, h \in X$, *if* $g^{-1}(0) \subsetneq h^{-1}(0)$ *then* $g^{-1}(1) \cap h^{-1}(0) \neq \emptyset$;

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(R) (i) *for all $f, h \in X$, if $f^{-1}(1) \subsetneq h^{-1}(1)$ then $h^{-1}(1) \cap f^{-1}(0) \neq \emptyset$;*
(ii) *for all $g, h \in X$, if $g^{-1}(0) \subsetneq h^{-1}(0)$ then $g^{-1}(1) \cap h^{-1}(0) \neq \emptyset$;*

Hence we may rewrite the conditions (S) and (R) above, and define them for any graph $\mathbb{X} = (X, R)$, as follows:

(S) for every $x, y \in X$, if $x \neq y$ then $xR \neq yR$ or $Rx \neq Ry$;

(R) (i) for all $x, z \in X$, if $zR \subsetneq xR$ then $(z, x) \notin R$;
(ii) for all $y, z \in X$, if $Rz \subsetneq Ry$ then $(y, z) \notin R$;

The (Ti) property

Let $\mathbb{X} = (X, R)$ be a graph and consider the following property:

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When R is reflexive, then (Ti) is equivalent to:

(Ti)′ for all $x, y \in X$, if $(x, y) \in R$, then there exists z such that $(x, z) \in R$ and $(z, y) \in R$ and for every $w \in X$,
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- If R were a partial order we would say that the elements z were in the interval $[x, y]$.
- For the elements z we will use the term transitive interval elements (with respect to $(x, y) \in R$).

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The (Ti) for frames is motivated by the (Ti) for graphs (paper).

TiRS graphs and frames (Craig, Gouveia, MH, 2015)

Definition

A TiRS graph (frame) is a reflexive graph (frame) that satisfies conditions (R), (S) and (Ti), i.e., it is a reflexive RS graph (RS frame) that satisfies condition (Ti).

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Proposition

For any bounded lattice $\mathbf{L}$*,*

(i) *its Ploščica's dual* $\mathrm{D}^{\flat}(\mathbf{L}) = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R)$ *is a TiRS graph;*

(ii) *the frame* $\mathbb{F}(\mathbf{L}) = (\mathrm{Filt}_M(\mathbf{L}), \mathrm{Idl}_M(\mathbf{L}), R)$ *is a TiRS frame.*

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Theorem

Let $\mathbb{X} = (X, R)$ *be a TiRS graph and* $\mathbb{F} = (X_1, X_2, R)$ *be a TiRS frame. There is a one-to-one correspondence between TiRS graphs and TiRS frames. (Details in our paper.)*

Example of an RS graph which is not TiRS

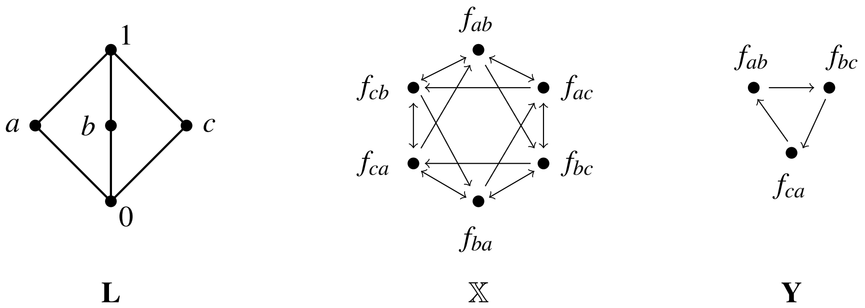
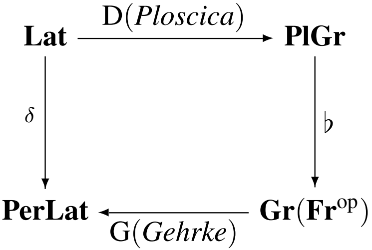


Figure : The graph $\mathbf{Y}$ is an RS graph which is not TiRS.

Ploščica–Gehrke in tandem (Craig, Gouveia, MH, 2015)



Theorem

Let $\mathbf{L}$ *be a bounded lattice and* $\mathbb{X} = \mathrm{D}^{\flat}(\mathbf{L})$ *be its dual (Ploščica's) TiRS graph. Let* $\rho(\mathbb{X})$ *be the frame associated to* $\mathbb{X}$ *and* $\mathrm{G}(\rho(\mathbb{X}))$ *be its corresponding (Gehrke's) perfect lattice of Galois-closed sets.*

CJIs, CMIs (Craig, Gouveia, MH, 2015)

Theorem

Let $\mathbb{X} = (X, E)$ *be a reflexive (R) graph (in particular a TiRS graph). Then* **the lattice** $\mathrm{C}(\mathbb{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbb{X}, \underset{\sim}{2})$ **is a perfect lattice**. *More precisely, for every* $\varphi \in \mathrm{C}(\mathbb{X})$*, we have*

$$\varphi = \bigvee\{\, J_x \mid J_x \leqslant \varphi \,\} \quad \text{and} \quad \varphi = \bigwedge\{\, M_y \mid \varphi \leqslant M_y \,\} \quad \text{and}$$

$$J^\infty(\mathrm{C}(\mathbb{X})) = \{\, J_x \mid x \in X \,\} \quad \text{and} \quad M^\infty(\mathrm{C}(\mathbb{X})) = \{\, M_y \mid y \in X \,\}$$

where the partial maps $J_x, M_y : X \to 2$ *are given by:*

$$J_x(z) = \begin{cases} 1 & \text{if } zE \subseteq xE \\ 0 & \text{if } (x,z) \notin E \\ - & \text{otherwise} \end{cases} \quad \text{and} \quad M_y(z) = \begin{cases} 1 & \text{if } (z,y) \notin E \\ 0 & \text{if } Ez \subseteq Ey \\ - & \text{otherwise.} \end{cases}$$

Question

Is every TiRS graph $\mathbb{X} = (X, R)$ **of the form** $\mathrm{D}^{\flat}(\mathbf{L}) = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R)$ *for some bounded lattice* $\mathbf{L}$*?*

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Answer

No. Every poset is a TiRS graph. A poset is said to be **representable** *if it is the underlying poset of some Priestley space and hence the untopologized dual of some bounded distributive lattice. It is known that* **non-representable posets exist** *and hence non-representable TiRS graphs exist.*

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Problem

Which TiRS graphs arise as duals of bounded lattices?

General. Birkhoff (Craig, Gouveia, MH, 2015)

Theorem

Every finite RS frame is a TiRS frame.

General. Birkhoff (Craig, Gouveia, MH, 2015)

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Theorem

There exists a dual representation of arbitrary finite lattices via finite TiRS graphs generalizing Birkhoff's representation.

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Proof.

Finite lattices = Finite perfect lattices

↕

Finite RS frames = Finite TiRS frames

↕

Finite TiRS graphs