A categorical structure of realizers for the Minimalist Foundation

S.Maschio (joint work with M.E.Maietti)

Department of Mathematics
University of Padua

TACL 2015
Ischia

The Minimalist Foundation

Many foundations in (constructive) mathematics...

The Minimalist Foundation

Many foundations in (constructive) mathematics...

$$\Downarrow$$

The Minimalist Foundation

Many foundations in (constructive) mathematics...

$$\Downarrow$$

Necessity of a common core:

The Minimalist Foundation

Many foundations in (constructive) mathematics...

$$\Downarrow$$

Necessity of a common core:

the minimalist foundation!

The Minimalist Foundation

Many foundations in (constructive) mathematics...

$$\Downarrow$$

Necessity of a common core:

the minimalist foundation!

(Maietti, Sambin 2005)

The formal system of the Minimalist Foundation (Maietti (2009)):

The formal system of the Minimalist Foundation (Maietti (2009)):

- 2-level theory based on versions of Martin Löf Type Theory;

The formal system of the Minimalist Foundation (Maietti (2009)):

- 2-level theory based on versions of Martin Löf Type Theory;
- an **intensional** level (**mTT**): computational content of proofs;

The formal system of the Minimalist Foundation (Maietti (2009)):

- 2-level theory based on versions of Martin Löf Type Theory;
- an **intensional** level (**mTT**): computational content of proofs;
- an **extensional** level (**emTT**): where to develop ordinary mathematics.

The intensional level has four sorts of types:

The intensional level has four sorts of types:

- *sets*: basic N_0, N_1 and constructors Π, Σ, $+$ and List, all *small propositions*;

The intensional level has four sorts of types:

- *sets*: basic $\mathsf{N}_0, \mathsf{N}_1$ and constructors Π, Σ, $+$ and List, all *small propositions*;
- *collections*: all *sets* and *propositions*, the collection of (codes for) small propositions prop_s, $A \to \mathsf{prop}_s$ with A set, constructor Σ;

The intensional level has four sorts of types:

- *sets*: basic $\mathsf{N}_0, \mathsf{N}_1$ and constructors Π, Σ, $+$ and List, all *small propositions*;
- *collections*: all *sets* and *propositions*, the collection of (codes for) small propositions prop_s, $A \to \mathsf{prop}_s$ with A set, constructor Σ;
- *propositions*: $\perp$ and closed under connectives $\wedge, \vee, \to$, collection bounded quantifiers and Id in collections.

The intensional level has four sorts of types:

- *sets*: basic $\mathsf{N}_0, \mathsf{N}_1$ and constructors Π, Σ, $+$ and List, all *small propositions*;
- *collections*: all *sets* and *propositions*, the collection of (codes for) small propositions prop_s, $A \rightarrow \mathsf{prop}_s$ with A set, constructor Σ;
- *propositions*: $\perp$ and closed under connectives $\wedge, \vee, \rightarrow$, collection bounded quantifiers and Id in collections.
- *small propositions* are like *propositions* with only set bounded quantifiers and Id relative to sets, it contains decodings $\tau(p)$ for $p \in \mathsf{prop}_s$.

emTT= extensional version of **mTT**

(includes extensionality of functions)

emTT= extensional version of **mTT**

(includes extensionality of functions)

+ power-collections of sets

emTT= extensional version of **mTT**

(includes extensionality of functions)

+ power-collections of sets

+ effective quotient sets

emTT= extensional version of **mTT**

(includes extensionality of functions)

+ power-collections of sets

+ effective quotient sets

+ proof-irrelevance of propositions

emTT= extensional version of **mTT**

(includes extensionality of functions)

+ power-collections of sets

+ effective quotient sets

+ proof-irrelevance of propositions

it is interpreted in **mTT** via a quotient completion

Realizability

Constructive mathematics =

Realizability

Constructive mathematics =
implicit computational mathematics =

Realizability

Constructive mathematics =
implicit computational mathematics =
abstract mathematics with computational interpretation

Realizability

Constructive mathematics =
implicit computational mathematics =
abstract mathematics with computational interpretation (**realizability**)

Kleene's realizability interpretation of connectives and quantifiers (for Heyting arithmetic)

Kleene's realizability interpretation of connectives and quantifiers (for Heyting arithmetic)

$r \Vdash t = s$ is $t = s \wedge r = 0$

Kleene's realizability interpretation of connectives and quantifiers (for Heyting arithmetic)

$r \Vdash t = s$ is $t = s \wedge r = 0$

$r \Vdash \phi \wedge \psi$ is $p_1(r) \Vdash \phi \wedge p_2(r) \Vdash \psi$

Kleene's realizability interpretation of connectives and quantifiers (for Heyting arithmetic)

$r \Vdash t = s$ is $t = s \wedge r = 0$

$r \Vdash \phi \wedge \psi$ is $p_1(r) \Vdash \phi \wedge p_2(r) \Vdash \psi$

$r \Vdash \phi \vee \psi$ is $(p_1(r) = 0 \wedge p_2(r) \Vdash \phi) \vee (p_1(r) = 1 \wedge p_2(r) \Vdash \psi)$

Kleene's realizability interpretation of connectives and quantifiers (for Heyting arithmetic)

$$r \Vdash t = s \text{ is } t = s \wedge r = 0$$
$$r \Vdash \phi \wedge \psi \text{ is } p_1(r) \Vdash \phi \wedge p_2(r) \Vdash \psi$$
$$r \Vdash \phi \vee \psi \text{ is } (p_1(r) = 0 \wedge p_2(r) \Vdash \phi) \vee (p_1(r) = 1 \wedge p_2(r) \Vdash \psi)$$
$$r \Vdash \phi \to \psi \text{ is } \forall x (x \Vdash \phi \to \{r\}(x) \Vdash \psi)$$

Kleene's realizability interpretation of connectives and quantifiers (for Heyting arithmetic)

$$r \Vdash t = s \text{ is } t = s \wedge r = 0$$

$$r \Vdash \phi \wedge \psi \text{ is } p_1(r) \Vdash \phi \wedge p_2(r) \Vdash \psi$$

$$r \Vdash \phi \vee \psi \text{ is } (p_1(r) = 0 \wedge p_2(r) \Vdash \phi) \vee (p_1(r) = 1 \wedge p_2(r) \Vdash \psi)$$

$$r \Vdash \phi \to \psi \text{ is } \forall x (x \Vdash \phi \to \{r\}(x) \Vdash \psi)$$

$$r \Vdash \exists x \phi(x) \text{ is } p_2(x) \Vdash \phi(p_1(x))$$

Kleene's realizability interpretation of connectives and quantifiers (for Heyting arithmetic)

$r \Vdash t = s$ is $t = s \wedge r = 0$

$r \Vdash \phi \wedge \psi$ is $p_1(r) \Vdash \phi \wedge p_2(r) \Vdash \psi$

$r \Vdash \phi \vee \psi$ is $(p_1(r) = 0 \wedge p_2(r) \Vdash \phi) \vee (p_1(r) = 1 \wedge p_2(r) \Vdash \psi)$

$r \Vdash \phi \to \psi$ is $\forall x(x \Vdash \phi \to \{r\}(x) \Vdash \psi)$

$r \Vdash \exists x \phi(x)$ is $p_2(x) \Vdash \phi(p_1(x))$

$r \Vdash \forall x \phi(x)$ is $\forall x(\{r\}(x) \Vdash \phi(x))$

Kleene's realizability interpretation of connectives and quantifiers (for Heyting arithmetic)

$$r \Vdash t = s \text{ is } t = s \wedge r = 0$$
$$r \Vdash \phi \wedge \psi \text{ is } p_1(r) \Vdash \phi \wedge p_2(r) \Vdash \psi$$
$$r \Vdash \phi \vee \psi \text{ is } (p_1(r) = 0 \wedge p_2(r) \Vdash \phi) \vee (p_1(r) = 1 \wedge p_2(r) \Vdash \psi)$$
$$r \Vdash \phi \rightarrow \psi \text{ is } \forall x(x \Vdash \phi \rightarrow \{r\}(x) \Vdash \psi)$$
$$r \Vdash \exists x\phi(x) \text{ is } p_2(x) \Vdash \phi(p_1(x))$$
$$r \Vdash \forall x\phi(x) \text{ is } \forall x(\{r\}(x) \Vdash \phi(x))$$

mTT extends **HA**.

Kleene's realizability interpretation of connectives and quantifiers (for Heyting arithmetic)

$$r \Vdash t = s \text{ is } t = s \wedge r = 0$$
$$r \Vdash \phi \wedge \psi \text{ is } p_1(r) \Vdash \phi \wedge p_2(r) \Vdash \psi$$
$$r \Vdash \phi \vee \psi \text{ is } (p_1(r) = 0 \wedge p_2(r) \Vdash \phi) \vee (p_1(r) = 1 \wedge p_2(r) \Vdash \psi)$$
$$r \Vdash \phi \to \psi \text{ is } \forall x(x \Vdash \phi \to \{r\}(x) \Vdash \psi)$$
$$r \Vdash \exists x \phi(x) \text{ is } p_2(x) \Vdash \phi(p_1(x))$$
$$r \Vdash \forall x \phi(x) \text{ is } \forall x(\{r\}(x) \Vdash \phi(x))$$

mTT extends **HA**.
We model **mTT** by extending Kleene realizability interpretation.

$\widehat{\mathrm{ID}}_1$

1. Theory of **Inductive definitions**:

$\widehat{\mathrm{ID}}_1$

1. Theory of **Inductive definitions**:
   consists of Peano Arithmetic

$\widehat{\mathsf{ID}}_1$

1. Theory of **Inductive definitions**:
   consists of Peano Arithmetic
   + fix points for positive paremeter-free operators

$\widehat{\mathrm{ID}}_1$

1. Theory of **Inductive definitions**:

   consists of Peano Arithmetic

   + fix points for positive paremeter-free operators

2. Same proof theoretical strength as first-order Martin Löf type theory with one universe

$\widehat{\mathrm{ID}}_1$

1. Theory of **Inductive definitions**:

   consists of Peano Arithmetic

   + fix points for positive paremeter-free operators

2. Same proof theoretical strength as first-order Martin Löf type theory with one universe

Collections of $\widehat{\mathrm{ID}}_1$

$$A = (|A|, \sim_A)$$

Collections of $\widehat{\mathrm{ID}}_1$

$$A = (|A|, \sim_A)$$

$|A| = \{x | \, \phi_A(x)\}$ is a (first-order) definable class of $\widehat{\mathrm{ID}_1}$

Collections of $\widehat{\mathrm{ID}}_1$

$$A = (|A|, \sim_A)$$

$|A| = \{x \mid \phi_A(x)\}$ is a (first-order) definable class of $\widehat{\mathrm{ID}_1}$

$x \varepsilon A$ is an abbreviation for $\phi_A(x)$.

Collections of $\widehat{\mathrm{ID}}_1$

$$A = (|A|, \sim_A)$$

$|A| = \{x |\, \phi_A(x)\}$ is a (first-order) definable class of $\widehat{\mathrm{ID}_1}$

$x \varepsilon A$ is an abbreviation for $\phi_A(x)$.

$x \sim_A y$ is a (first-order) $\widehat{\mathrm{ID}_1}$-definable equivalence relation on $|A|$

Operations

An arrow from A to B is an operation (a computable function):

Operations

An arrow from A to B is an operation (a computable function):
NOT a function (a single valued relation)!!!

Operations

An arrow from A to B is an operation (a computable function):
NOT a function (a single valued relation)!!!

$$[\mathbf{n}]_{\equiv_{A,B}} : A \to B$$

Operations

An arrow from A to B is an operation (a computable function):
NOT a function (a single valued relation)!!!

$$[\mathbf{n}]_{\equiv_{A,B}} : A \to B$$

n is a numeral

Operations

An arrow from A to B is an operation (a computable function):
NOT a function (a single valued relation)!!!

$$[\mathbf{n}]_{\equiv_{A,B}} : A \to B$$

$\mathbf{n}$ is a numeral

$$x \sim_A y \vdash_{\widehat{\mathrm{ID}}_1} \{\mathbf{n}\}(x) \sim_B \{\mathbf{n}\}(y)$$

Operations

An arrow from A to B is an operation (a computable function):
NOT a function (a single valued relation)!!!

$$[\mathbf{n}]_{\equiv_{A,B}} : A \to B$$

n is a numeral

$$x \sim_A y \vdash_{\widehat{\mathsf{ID}}_1} \{\mathbf{n}\}(x) \sim_B \{\mathbf{n}\}(y)$$

$$\mathbf{n} \equiv_{A,B} \mathbf{m} \text{ if and only if } x \varepsilon A \vdash_{\widehat{\mathsf{ID}}_1} \{\mathbf{n}\}(x) \sim_B \{\mathbf{m}\}(x)$$

Proof irrelevance

$$\text{for all } A, f, g \qquad A \underset{g}{\overset{f}{\rightrightarrows}} P \;\Rightarrow f = g$$

Proof irrelevance

$$\text{for all } A, f, g \qquad A \overset{f}{\underset{g}{\rightrightarrows}} P \;\Rightarrow f = g$$

equivalent to

Proof irrelevance

$$\text{for all } A, f, g \qquad A \underset{g}{\overset{f}{\rightrightarrows}} P \Rightarrow f = g$$

equivalent to

$$x \varepsilon P \wedge y \varepsilon P \vdash_{\widehat{\mathsf{ID}}_1} x \sim_P y$$

Proof irrelevance

$$\text{for all } A, f, g \qquad A \underset{g}{\overset{f}{\rightrightarrows}} P \;\Rightarrow f = g$$

equivalent to

$$x \varepsilon P \wedge y \varepsilon P \vdash_{\widehat{\mathrm{ID}}_1} x \sim_P y$$

Propositions of $\widehat{\mathrm{ID}}_1 =^{def}$ proof-irrelevant collections of $\widehat{\mathrm{ID}}_1$

Proof irrelevance

$$\text{for all } A, f, g \qquad A \underset{g}{\overset{f}{\rightrightarrows}} P \;\Rightarrow f = g$$

equivalent to

$$x \varepsilon P \wedge y \varepsilon P \vdash_{\widehat{\mathrm{ID}}_1} x \sim_P y$$

Propositions of $\widehat{\mathrm{ID}}_1 =^{def}$ proof-irrelevant collections of $\widehat{\mathrm{ID}}_1$

Propositions $\equiv$ trivial quotients of the collections of their realizers

Universes

An extended realizability interpretation of sets and small propositions of **mTT** can be encoded using fix points.

Universes

An extended realizability interpretation of sets and small propositions of **mTT** can be encoded using fix points.

$$\Downarrow$$

collections of $\widehat{\mathrm{ID}_1}$ of (codes for) sets and small propositions of **mTT**: US and USP.

Universes

An extended realizability interpretation of sets and small propositions of **mTT** can be encoded using fix points.

$$\Downarrow$$

collections of $\widehat{\mathrm{ID}}_1$ of (codes for) sets and small propositions of **mTT**: US and USP.

$$f = [\mathbf{a}]_{\equiv} : 1 \to \mathrm{US}(\mathrm{P})$$

is an operation

$$\Downarrow$$

$$\tau(f) := \big(\{x|\, x \overline{\varepsilon} \{\mathbf{a}\}(0)\}\, ,\, x \equiv_{\{\mathbf{a}\}(0)} y\big)$$

is a collection (proposition).

Universes

An extended realizability interpretation of sets and small propositions of **mTT** can be encoded using fix points.

$$\Downarrow$$

collections of $\widehat{\mathrm{ID}}_1$ of (codes for) sets and small propositions of **mTT**: US and USP.

$$f = [\mathbf{a}]_{\equiv} : 1 \to \mathrm{US}(\mathrm{P})$$

is an operation

$$\Downarrow$$

$$\tau(f) := \left(\{x \mid x \overline{\varepsilon} \{\mathbf{a}\}(0)\}\,,\, x \equiv_{\{\mathbf{a}\}(0)} y\right)$$

is a collection (proposition).

A set (small proposition) of $\widehat{\mathrm{ID}}_1$ is a collection (proposition) of $\widehat{\mathrm{ID}}_1$ of the form $\tau(f)$ for $f : 1 \to \mathrm{US}(P)$.

first summary

Obtain a commutative diagram in **Cat**

first summary

Obtain a commutative diagram in **Cat**

$$
\begin{array}{ccc}
\mathcal{S} & \hookrightarrow & \mathcal{C} \\
\uparrow & & \uparrow \\
\mathcal{P}_s & \hookrightarrow & \mathcal{P}
\end{array}
$$

Contexts

Define

Contexts

Define

a category of contexts of $\widehat{\mathrm{ID}}_1$, **Cont**

Contexts

Define

- a category of contexts of $\widehat{\mathrm{ID}}_1$, **Cont**
- a functor **Col** : $\mathbf{Cont}^{op} \to \mathbf{Cat}$

Contexts

Define

- a category of contexts of $\widehat{\mathrm{ID}}_1$, **Cont**
- a functor **Col** : $\mathbf{Cont}^{op} \to \mathbf{Cat}$
- in such a way that if Γ is in **Cont**:

Contexts

Define

a category of contexts of $\widehat{\mathrm{ID}}_1$, **Cont**

a functor **Col** : $\mathbf{Cont}^{op} \to \mathbf{Cat}$

in such a way that if Γ is in **Cont**:

$$A \in \mathbf{Col}(\Gamma) \Leftrightarrow [\Gamma, A] \in \mathbf{Cont}$$

Contexts

Define

a category of contexts of $\widehat{\mathrm{ID}}_1$, **Cont**

a functor $\mathbf{Col} : \mathbf{Cont}^{op} \to \mathbf{Cat}$

in such a way that if Γ is in **Cont**:

$$A \in \mathbf{Col}(\Gamma) \Leftrightarrow [\Gamma, A] \in \mathbf{Cont}$$

$$f : A \to B \text{ in } \mathbf{Col}(\Gamma) \Leftrightarrow f : \mathbf{pr}_{[\Gamma,A]} \to \mathbf{pr}_{[\Gamma,B]} \text{ in } \mathbf{Cont}/\Gamma$$

Contexts

Define

a category of contexts of $\widehat{\mathrm{ID}}_1$, **Cont**

a functor **Col** : $\mathbf{Cont}^{op} \to \mathbf{Cat}$

in such a way that if Γ is in **Cont**:

$$A \in \mathbf{Col}(\Gamma) \Leftrightarrow [\Gamma, A] \in \mathbf{Cont}$$

$$f : A \to B \text{ in } \mathbf{Col}(\Gamma) \Leftrightarrow f : \mathbf{pr}_{[\Gamma,A]} \to \mathbf{pr}_{[\Gamma,B]} \text{ in } \mathbf{Cont}/\Gamma$$

where $\mathbf{pr}_{[\Gamma,A]}$ is the projection from $[\Gamma, A]$ to Γ.

Contexts

Define

a category of contexts of $\widehat{\mathrm{ID}}_1$, **Cont**

a functor **Col** : $\mathbf{Cont}^{op} \to \mathbf{Cat}$

in such a way that if Γ is in **Cont**:

$$A \in \mathbf{Col}(\Gamma) \Leftrightarrow [\Gamma, A] \in \mathbf{Cont}$$

$$f : A \to B \text{ in } \mathbf{Col}(\Gamma) \Leftrightarrow f : \mathbf{pr}_{[\Gamma,A]} \to \mathbf{pr}_{[\Gamma,B]} \text{ in } \mathbf{Cont}/\Gamma$$

where $\mathbf{pr}_{[\Gamma,A]}$ is the projection from $[\Gamma, A]$ to Γ.

We define appropriate functors **Set**, **Prop**, $\mathbf{Prop}_s$.

second summary

$$\begin{array}{ccc} \mathbf{Set} & \hookrightarrow & \mathbf{Col} \\ \uparrow & & \uparrow \\ \mathbf{Prop}_s & \hookrightarrow & \mathbf{Prop} \end{array} \qquad \text{in } \mathbf{Cat}^{\mathbf{Cont}^{op}}$$

second summary

$$\begin{array}{ccc} \mathbf{Set} & \hookrightarrow & \mathbf{Col} \\ \uparrow & & \uparrow \\ \mathbf{Prop}_s & \hookrightarrow & \mathbf{Prop} \end{array} \qquad \text{in } \mathbf{Cat}^{\mathbf{Cont}^{op}}$$

$$\mathrm{USP}_{cat} \hookrightarrow \mathrm{US}_{cat} \qquad \text{in } \mathbf{Cat}(\mathcal{C})$$

second summary

$$\begin{array}{ccc} \mathbf{Set} & \hookrightarrow & \mathbf{Col} \\ \uparrow & & \uparrow \\ \mathbf{Prop}_s & \hookrightarrow & \mathbf{Prop} \end{array} \qquad \text{in } \mathbf{Cat}^{\mathbf{Cont}^{op}}$$

$$\mathrm{USP}_{cat} \hookrightarrow \mathrm{US}_{cat} \qquad \text{in } \mathbf{Cat}(\mathcal{C})$$

$$\begin{array}{ccc} \mathcal{S} \equiv \mathbf{Set}([\,]) \equiv \Gamma(\mathrm{US}_{cat}) & \hookrightarrow & \mathcal{C} \equiv \mathbf{Col}([\,]) \equiv \mathbf{Cont} \\ \uparrow & & \uparrow \\ \mathcal{P}_s \equiv \mathbf{Prop}_s([\,]) \equiv \Gamma(\mathrm{USP}_{cat}) & \hookrightarrow & \mathcal{P} \equiv \mathbf{Prop}([\,]) \end{array} \qquad \text{in } \mathbf{Cat}$$

Properties

$\mathbf{Col}(\Gamma)$ and $\mathbf{Set}(\Gamma)$ are cartesian closed finitely complete with list objects and stable disjoint finite coproducts;

Properties

$\mathbf{Col}(\Gamma)$ and $\mathbf{Set}(\Gamma)$ are cartesian closed finitely complete with list objects and stable disjoint finite coproducts;

$\mathbf{Col}_{\mathbf{pr}_{[\Gamma,A]}} : \mathbf{Col}(\Gamma) \to \mathbf{Col}([\Gamma, A])$ has left and right adjoints;

Properties

$\mathbf{Col}(\Gamma)$ and $\mathbf{Set}(\Gamma)$ are cartesian closed finitely complete with list objects and stable disjoint finite coproducts;

$\mathbf{Col}_{\mathbf{pr}_{[\Gamma,A]}} : \mathbf{Col}(\Gamma) \to \mathbf{Col}([\Gamma, A])$ has left and right adjoints;

$\mathbf{Col}(\Gamma)$ contains an universe object of small propositions in context;

Properties

$\mathbf{Col}(\Gamma)$ and $\mathbf{Set}(\Gamma)$ are cartesian closed finitely complete with list objects and stable disjoint finite coproducts;

$\mathbf{Col}_{\mathbf{pr}_{[\Gamma,A]}} : \mathbf{Col}(\Gamma) \to \mathbf{Col}([\Gamma, A])$ has left and right adjoints;

$\mathbf{Col}(\Gamma)$ contains an universe object of small propositions in context;

$\mathbf{Set}_{\mathbf{pr}_{[\Gamma,A]}} : \mathbf{Set}(\Gamma) \to \mathbf{Set}([\Gamma, A])$ has left and right adjoints if $A \in \mathbf{Set}([\Gamma])$;

Properties

$\mathbf{Col}(\Gamma)$ and $\mathbf{Set}(\Gamma)$ are cartesian closed finitely complete with list objects and stable disjoint finite coproducts;

$\mathbf{Col}_{\mathbf{pr}_{[\Gamma,A]}} : \mathbf{Col}(\Gamma) \to \mathbf{Col}([\Gamma, A])$ has left and right adjoints;

$\mathbf{Col}(\Gamma)$ contains an universe object of small propositions in context;

$\mathbf{Set}_{\mathbf{pr}_{[\Gamma,A]}} : \mathbf{Set}(\Gamma) \to \mathbf{Set}([\Gamma, A])$ has left and right adjoints if $A \in \mathbf{Set}([\Gamma])$;

Prop is a first-order hyperdoctrine;

Properties

$\mathbf{Col}(\Gamma)$ and $\mathbf{Set}(\Gamma)$ are cartesian closed finitely complete with list objects and stable disjoint finite coproducts;

$\mathbf{Col}_{\mathbf{pr}_{[\Gamma, A]}} : \mathbf{Col}(\Gamma) \to \mathbf{Col}([\Gamma, A])$ has left and right adjoints;

$\mathbf{Col}(\Gamma)$ contains an universe object of small propositions in context;

$\mathbf{Set}_{\mathbf{pr}_{[\Gamma, A]}} : \mathbf{Set}(\Gamma) \to \mathbf{Set}([\Gamma, A])$ has left and right adjoints if $A \in \mathbf{Set}([\Gamma])$;

Prop is a first-order hyperdoctrine;

$\mathbf{Prop}_s$ is a doctrine with left and right adjoint for
$\mathbf{Prop}_{s, \mathbf{pr}_{[\Gamma, A]}} : \mathbf{Prop}_s(\Gamma) \to \mathbf{Prop}_s([\Gamma, A])$ with $A \in \mathbf{Set}([\Gamma])$;

(Partial) interpretation

of (fully annotated!) syntax of **mTT**:

(Partial) interpretation

of (fully annotated!) syntax of **mTT**:

Contexts $\Gamma \mapsto$ objects $\mathcal{I}(\Gamma)$ of **Cont**;

(Partial) interpretation

of (fully annotated!) syntax of **mTT**:

Contexts $\Gamma \mapsto$ objects $\mathcal{I}(\Gamma)$ of **Cont**;

Collections A in context $\Gamma \mapsto$ objects of $\mathbf{Col}(\mathcal{I}(\Gamma))$;

(Partial) interpretation

of (fully annotated!) syntax of **mTT**:

Contexts $\Gamma \mapsto$ objects $\mathcal{I}(\Gamma)$ of **Cont**;

Collections A in context $\Gamma \mapsto$ objects of $\mathbf{Col}(\mathcal{I}(\Gamma))$;

Sets A in context $\Gamma \mapsto$ objects of $\mathbf{Set}(\mathcal{I}(\Gamma))$;

(Partial) interpretation

of (fully annotated!) syntax of **mTT**:

Contexts $\Gamma \mapsto$ objects $\mathcal{I}(\Gamma)$ of **Cont**;

Collections A in context $\Gamma \mapsto$ objects of $\mathbf{Col}(\mathcal{I}(\Gamma))$;

Sets A in context $\Gamma \mapsto$ objects of $\mathbf{Set}(\mathcal{I}(\Gamma))$;

Propositions A in context $\Gamma \mapsto$ objects of $\mathbf{Prop}(\mathcal{I}(\Gamma))$;

(Partial) interpretation

of (fully annotated!) syntax of **mTT**:

- Contexts $\Gamma \mapsto$ objects $\mathcal{I}(\Gamma)$ of **Cont**;
- Collections A in context $\Gamma \mapsto$ objects of $\mathbf{Col}(\mathcal{I}(\Gamma))$;
- Sets A in context $\Gamma \mapsto$ objects of $\mathbf{Set}(\mathcal{I}(\Gamma))$;
- Propositions A in context $\Gamma \mapsto$ objects of $\mathbf{Prop}(\mathcal{I}(\Gamma))$;
- Small propositions A in context $\Gamma \mapsto$ objects of $\mathbf{Prop}_s(\mathcal{I}(\Gamma))$;

(Partial) interpretation

of (fully annotated!) syntax of **mTT**:

Contexts $\Gamma \mapsto$ objects $\mathcal{I}(\Gamma)$ of **Cont**;

Collections A in context $\Gamma \mapsto$ objects of $\mathbf{Col}(\mathcal{I}(\Gamma))$;

Sets A in context $\Gamma \mapsto$ objects of $\mathbf{Set}(\mathcal{I}(\Gamma))$;

Propositions A in context $\Gamma \mapsto$ objects of $\mathbf{Prop}(\mathcal{I}(\Gamma))$;

Small propositions A in context $\Gamma \mapsto$ objects of $\mathbf{Prop}_s(\mathcal{I}(\Gamma))$;

Elements a in context $\Gamma \mapsto$ global elements in $\mathbf{Col}(\mathcal{I}(\Gamma))$.

(Partial) interpretation

of (fully annotated!) syntax of **mTT**:

Contexts $\Gamma \mapsto$ objects $\mathcal{I}(\Gamma)$ of **Cont**;
Collections A in context $\Gamma \mapsto$ objects of $\mathbf{Col}(\mathcal{I}(\Gamma))$;
Sets A in context $\Gamma \mapsto$ objects of $\mathbf{Set}(\mathcal{I}(\Gamma))$;
Propositions A in context $\Gamma \mapsto$ objects of $\mathbf{Prop}(\mathcal{I}(\Gamma))$;
Small propositions A in context $\Gamma \mapsto$ objects of $\mathbf{Prop}_s(\mathcal{I}(\Gamma))$;
Elements a in context $\Gamma \mapsto$ global elements in $\mathbf{Col}(\mathcal{I}(\Gamma))$.

Validity

1. Every judgment J of **mTT** for which **mTT** $\vdash J$ is validated by the realizability model $\mathcal{R}$

Validity

1. Every judgment J of **mTT** for which $\mathbf{mTT} \vdash J$ is validated by the realizability model $\mathcal{R}$
2. $\mathcal{R}$ validates **CT**

Validity

1. Every judgment J of **mTT** for which $\mathbf{mTT} \vdash J$ is validated by the realizability model $\mathcal{R}$
2. $\mathcal{R}$ validates **CT**
3. $\mathcal{R}$ validates $\mathbf{AC_N}$ and $\mathbf{AC_!}$

Validity

1. Every judgment J of **mTT** for which $\mathbf{mTT} \vdash J$ is validated by the realizability model $\mathcal{R}$
2. $\mathcal{R}$ validates **CT**
3. $\mathcal{R}$ validates $\mathbf{AC_N}$ and $\mathbf{AC_!}$
4. from $\mathcal{R}$, by elementary quotient completion (Maietti, Rosolini), model of $\mathbf{emTT} + \mathbf{CT}$

Validity

1. Every judgment J of **mTT** for which $\mathbf{mTT} \vdash J$ is validated by the realizability model $\mathcal{R}$
2. $\mathcal{R}$ validates **CT**
3. $\mathcal{R}$ validates $\mathbf{AC_N}$ and $\mathbf{AC_!}$
4. from $\mathcal{R}$, by elementary quotient completion (Maietti, Rosolini), model of $\mathbf{emTT} + \mathbf{CT}$
5. validity of $\mathbf{AC_!}$ is not preserved by completion

Validity

1. Every judgment J of **mTT** for which $\mathbf{mTT} \vdash J$ is validated by the realizability model $\mathcal{R}$
2. $\mathcal{R}$ validates **CT**
3. $\mathcal{R}$ validates $\mathbf{AC_N}$ and $\mathbf{AC_!}$
4. from $\mathcal{R}$, by elementary quotient completion (Maietti, Rosolini), model of $\mathbf{emTT} + \mathbf{CT}$
5. validity of $\mathbf{AC_!}$ is not preserved by completion

Work in progress

Minimalist predicative version of

tripos-to-topos construction

Work in progress

Minimalist predicative version of

tripos-to-topos construction

Hyland's effective topos

Work in progress

Minimalist predicative version of

tripos-to-topos construction

Hyland's effective topos

realizability toposes

Work in progress

Minimalist predicative version of

tripos-to-topos construction

Hyland's effective topos

realizability toposes

References

this work: M.E.Maietti, S.Maschio, *A predicative realizability tripos for the Minimalist Foundations*, in preparation.

References

this work: M.E.Maietti, S.Maschio, *A predicative realizability tripos for the Minimalist Foundations*, in preparation.

1. Other papers on **Minimalist Foundations**
    1. F.C., *Toward a minimalist foundation for constructive mathematics*, **From Sets and Types to Topology and Analysis**, 2005.
    2. M.E.Maietti, *A minimalist two-level foundation for constructive mathematics*, **Annals of pure and applied logic**, 2009.
    3. M.E.Maietti, G.Sambin *Why topology in the Minimalist Foundation must be pointfree.*, **Logic and Logical Philosophy**, 2013.

References

this work: M.E.Maietti, S.Maschio, *A predicative realizability tripos for the Minimalist Foundations*, in preparation.

1. Other papers on **Minimalist Foundations**
    1. F.C., *Toward a minimalist foundation for constructive mathematics*, **From Sets and Types to Topology and Analysis**, 2005.
    2. M.E.Maietti, *A minimalist two-level foundation for constructive mathematics*, **Annals of pure and applied logic**, 2009.
    3. M.E.Maietti, G.Sambin *Why topology in the Minimalist Foundation must be pointfree.*, **Logic and Logical Philosophy**, 2013.
2. About **Realizability Models for Type Theory and Minimalist Foundation**
    1. M. Beeson, *Foundations of Constructive Mathematics*, Springer-Verlag, 1985.
    2. M.E.Maietti, S.Maschio *An extensional Kleene realizability semantics for the Minimalist Foundation*, to appear in **TYPES'14 post-proceedings**, 2015.