

A categorical structure of realizers for the Minimalist Foundation

S.Maschio (joint work with M.E.Maietti)



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Ischia

The Minimalist Foundation

Many foundations in (constructive) mathematics...

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Necessity of a common core:

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(Maietti, Sambin 2005)

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- an **intensional** level (**mTT**): computational content of proofs;
- an **extensional** level (**emTT**): where to develop ordinary mathematics.

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- *small propositions* are like *propositions* with only set bounded quantifiers and Id relative to sets, it contains decodings $\tau(p)$ for $p \in \text{prop}_s$.

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it is interpreted in **mTT** via a quotient completion

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We model **mTT** by extending Kleene realizability interpretation.

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$x \sim_A y$ is a (first-order) $\widehat{\text{ID}}_1$ -definable equivalence relation on $|A|$

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$\mathbf{n} \equiv_{A,B} \mathbf{m}$ if and only if $x \in A \vdash_{\widehat{\text{ID}}_1} \{\mathbf{n}\}(x) \sim_B \{\mathbf{m}\}(x)$

Proof irrelevance

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Propositions \equiv trivial quotients of the collections of their realizers

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is an operation



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A set (small proposition) of \widehat{ID}_1 is a collection (proposition) of \widehat{ID}_1 of the form $\tau(f)$ for $f : 1 \rightarrow \text{US}(P)$.

first summary

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$$\begin{array}{ccc} S^{\mathcal{C}} & \longrightarrow & \mathcal{C} \\ \uparrow & & \uparrow \\ \mathcal{P}_s^{\mathcal{C}} & \longrightarrow & \mathcal{P} \end{array}$$

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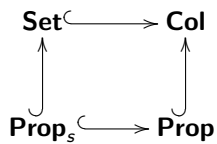
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We define appropriate functors **Set**, **Prop**, **Prop_s**.

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Work in progress

Minimalist predicative version of
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References

this work: M.E.Maietti, S.Maschio, *A predicative realizability tripos for the Minimalist Foundations*, in preparation.

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④ Other papers on **Minimalist Foundations**

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