K-theory of modules as model-theoretic structures

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- L: language, M: L-structure.
- Definable always means definable with parameters from M.
- $\overline{\mathrm{Def}}(M)$: collection of definable subsets of M^n, $n \geq 1$.
- Definable isomorphism is a bijection with definable graph.
- $[\,] : \overline{\mathrm{Def}}(M) \to \widetilde{\mathrm{Def}}(M)$ is the natural surjection.
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(Krajíček-Scanon)

The model-theoretic Grothendieck ring, $K_0(M)$, is the ring completion of the above semiring.

Examples of Grothendieck ring

- $K_0(\text{finite structure}) \cong \mathbb{Z}$.
- Cluckers-Haskell: $K_0(\mathbb{F}_q((t))) = 0$, $K_0(\mathbb{Q}_p) = 0$.
- Krajíček-Scanlon: $K_0(RCF) \cong \mathbb{Z}$, $\mathbb{Z}[X_i : i \in \mathfrak{c}] \subseteq K_0(\mathbb{C})$.
- Denef-Loeser: $K_0(\mathbb{C})$ admits $\mathbb{Z}[u, v]$ as a quotient.

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- If M, N are L-structures and $M \preceq N$ then $K_0(M) \preceq K_0(N)$.
- If $M \equiv N$, then $\widetilde{\mathrm{Def}}(M) \equiv_{\exists_1} \widetilde{\mathrm{Def}}(N)$ in L_{ring}. As the Grothendieck ring $K_0(M)$ is existentially interpretable in $\widetilde{\mathrm{Def}}(M)$, we have $K_0(M) \equiv_{\exists_1} K_0(N)$.

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Conjecture (Prest)

$\mathcal{R}$: unital ring, $L_{\mathcal{R}}$: language of right $\mathcal{R}$-modules, $M_{\mathcal{R}}$: right $\mathcal{R}$-module. If $M \neq 0$, then $K_0(M_{\mathcal{R}}) \neq 0$.

K-theory of Symmetric Monoidal Groupoids

- Groupoid: A category where every morphism has a two-sided inverse.
- Connected Groupoid: A groupoid $\mathcal{G}$ where $\mathcal{G}(A, B) \neq \emptyset \ \forall A, B \in \mathcal{G}$.

Proposition

In a connected groupoid $\mathcal{G}$, the group $\mathcal{G}(A, A)$ is in bijection with the set $\mathcal{G}(A, B)$ for all $A, B \in \mathcal{G}$.

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Definition (Quillen)

$$K_n : \mathrm{SMonGpd} \xrightarrow{S^{-1}S} \mathrm{SMonGpd} \xrightarrow{B} \mathrm{Top} \xrightarrow{\pi_n} \mathrm{Ab}$$

Theorem (Barratt-Priddy-Quillen-Segal)

The groups $K_n(\mathrm{FinSets}^{iso})$ *are the stable homotopy groups of spheres,* π_n^s.
In particular, $K_0(\mathrm{FinSets}^{iso}) = \mathbb{Z}$ *and* $K_1(\mathrm{FinSets}^{iso}) = \mathbb{Z}_2$.

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A pairing on $(S, \star, 1)$ is a functor $\otimes : S \times S \to S$ that bi-distributes over $\star$.

Theorem (Loday)

A pairing on S gives product maps $K_p(S) \otimes K_q(S) \to K_{p+q}(S)$.
In particular, $K_0(S)$ gets a ring structure.

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If $\mathcal{S}(M) := (\overline{\mathrm{Def}}(M)^{iso}, \sqcup, \times, \varnothing, \{*\})$, then $K_0(M) \cong K_0(\mathcal{S}(M))$.

Definition (K.)

For $n \geq 0$, $K_n(M) := K_n(\mathcal{S}(M))$; functorial on elementary embeddings.

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If M is finite, then $\mathcal{S}(M) \simeq \mathrm{FinSets}^{iso}$ and hence $K_n(M) = \pi_n^s$.

Model theory of modules

$\mathcal{R}$: unital ring, $L_{\mathcal{R}} = \langle 0, +, -, m_r : r \in \mathcal{R} \rangle$, M : right $\mathcal{R}$-module.

- A positive primitive (*pp*) formula is an $L_{\mathcal{R}}(M)$-formula of the form $\exists \overline{y}(\overline{x}\ \overline{y}\ \overline{a})G = 0$, where G is a finite matrix with entries in $\mathcal{R}$ and $\overline{a} \in M$.
- Every parameter free *pp*-formula defines a subgroup of M^n and a general *pp*-formula defines either the empty set or a coset of a *pp*-subgroup.

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Baur-Monk *pp*-elimination theorem

Theory of M is determined by the invariants of *pp*-pairs of subgroups of M. Every definable subset of M^n, $n \geq 1$ is a boolean combination of *pp*-definable sets.

Grothendieck ring of a module

Theorem (Perera)

Grothendieck ring is an invariant of the theory of the module.
If F is a field and V_F is a nonzero F-vector space, then $K_0(V_F) \cong \mathbb{Z}[x]$.

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Structure theorem for $K_0(M_{\mathcal{R}})$ (K.)

$$K_0(M_{\mathcal{R}}) \cong \mathbb{Z}[\mathcal{X}]/\mathcal{J};$$

$\mathcal{X}$: multiplicative monoid of colours (*pp*-isomorphism classes of *pp*-sets),
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Corollary

If $M \neq 0$, then there is a split embedding $\mathbb{Z} \to K_0(M)$.
In particular $K_0(M)$ is non-trivial proving Prest's conjecture.

Trivial Invariants Ideal: Real vector space $\mathbb{R}_{\mathbb{R}}$

- The *pp*-sets are points, lines, planes,$\cdots$
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Point	1
Line	x
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Sketch of the proof

Grothendieck ring has another presentations in terms of cut-and-paste relations.
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- Combine the data of global characteristics to form the monoid ring $\mathbb{Z}[\mathcal{X}]$.

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Conjecture (K.)

If $\mathcal{J}(M_\mathcal{R}) = 0$ and, for $\mathcal{A} \in \mathcal{X}$, $\mathrm{Aut}_{pp}(\mathcal{A})$ denotes the *pp*-automorphism group of any *pp*-set in $\mathcal{A}$, then $K_1(M_\mathcal{R}) = \bigoplus_{\mathcal{A}\in\mathcal{X}}(\mathbb{Z}_2 \oplus \mathrm{Aut}_{pp}(\mathcal{A})^{ab})$.

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THANK YOU!