IMPRECISE PROBABILITIES ON MV-ALGEBRAS REVISITED

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Uncertain reasoning for many-valued events

- events are identified with elements of some MV-algebra M
- we adopt a bookmaking scheme in which a bookmaker and a bettor wager money on events $a_1, \ldots, a_k \in M$
- the betting odds $t(a_1), \ldots, t(a_k) \in [0, 1]$ are coherent if there is no winning strategy for the bettor

Coherence theorems state the following equivalence

1. the book $t(a_1), \ldots, t(a_k) \in [0, 1]$ is coherent
2. t can be extended to a particular functional on M

GOAL

The underlying booking schemes determine the properties of t:

1. de Finetti: Boolean algebra M, t is a probability measure
2. Mundici: MV-algebra M, t is a state
3. Montagna et al.: t is an upper/lower state

We will look at the properties of lower states in detail.

AGENDA

MV-algebras and states

Lower states

Comonotonically additive lower states

MV-ALGEBRAS AND STATES

Definition (Chang, 1958)

An MV-algebra is an algebra $\langle M, \oplus, \neg, \mathbf{0}\rangle$ such that

- $\langle M, \oplus, \mathbf{0}\rangle$ is an abelian monoid
- $\neg\neg a = a$
- $a \oplus \neg\mathbf{0} = \neg\mathbf{0}$
- $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$

Derived operation	Definition
$\mathbf{1}$	$\neg\mathbf{0}$
$a \odot b$	$\neg(\neg a \oplus \neg b)$
$a \vee b$	$\neg(\neg a \oplus b) \oplus b$

The real unit interval $[0, 1]$ with the operations:

Operation	Definition
$\mathbf{0}$	0
$\mathbf{1}$	1
$a \oplus b$	$\min(a + b, 1)$
$a \odot b$	$\max(a + b - 1, 0)$
$\neg a$	$1 - a$

The total order is the usual $\leq$.

EXAMPLES OF MV-ALGEBRAS

Example

Any Boolean algebra:
$\oplus = \vee$, $\odot = \wedge$

Example

Family C of functions $X \to [0,1]$ with the pointwise operations:

- $\mathbf{0}, \mathbf{1} \in C$
- if $f \in C$, then $\neg f \in C$
- if $f, g \in C$, then $f \oplus g \in C$

Example

Let G be an abelian ℓ-group with the order unit $\mathbf{1}$.
The order interval

$$\Gamma(G) := [\mathbf{0}, \mathbf{1}]$$

is an MV-algebra with

$$a \oplus b = (a + b) \wedge \mathbf{1}$$
$$\neg a = \mathbf{1} - a$$

$M = \Gamma(G)$

Theorem (Mundici)

Γ is a categorical equivalence between

- the category of MV-algebras and
- the category of abelian ℓ-groups with an order unit

MV-algebra	ℓ-group
$\{0, 1\}$	$\langle \mathbb{Z}, 1 \rangle$
$[0, 1]^n$	$\langle \mathbb{R}^n, (1, \ldots, 1) \rangle$

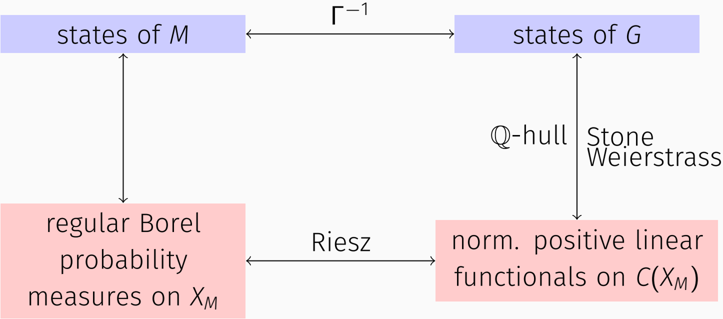
Definition

A state of an MV-algebra M is a functional $s\colon M \to [0,1]$ such that $s(1) = 1$ and

$$s(a \oplus b) = s(a) + s(b), \quad \text{whenever } a \odot b = 0.$$

Example

- finitely-additive probability on a Boolean algebra M
- valuation := MV-homomorphism $M \to [0,1]_{MV}$
- integral over MV-algebra of continuous functions



Contravariant functor $M \in \mathbf{MV} \mapsto X_M \in \mathbf{Comp}$

Proof by embedding $M \hookrightarrow G \hookrightarrow G(\mathbb{Q}) \hookrightarrow C(X_M)$

Certain bookmaking schemes do not lead to additivity:

M. Fedel, K. Keimel, F. Montagna, and W. Roth.
Imprecise probabilities, bets and functional analytic methods in Łukasiewicz logic.
Forum Mathematicum, 25(2):405–441, 2013.

LOWER STATES

Definition

Let G be an ℓ-group. A functional $t\colon G \to \mathbb{R}$ is a lower state if it is

1. **monotone**: $a \leq b$ implies $t(a) \leq t(b)$
2. **superadditive**: $t(a+b) \geq t(a) + t(b)$
3. **$\mathbb{N}$-homogeneous**: $t(na) = nt(a)$ for every $n \in \mathbb{N}$
4. **strongly normalized**: $t(1) = 1$ and $t(-1) = -1$

Example

Let $\mathcal{C} \neq \emptyset$ be a set of states of G and put

$$t_{\mathcal{C}}(a) := \inf\{s(a) \mid s \in \mathcal{C}\}, \qquad a \in G.$$

Definition

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Proposition

Let M be a 2-divisible MV-algebra. There is a bijection between

1. lower states of M
2. functionals $t\colon M \to [0,1]$ that are monotone, superadditive, $\mathbb{N}$-homogeneous and satisfy for every $a \in M$:

$$t\left(a \oplus \frac{1}{n}\mathbf{1}\right) = t(a) + \frac{1}{n}, \quad \text{if } a \odot \frac{1}{n}\mathbf{1} = 0 \text{ and } \mathbf{1} \text{ is divisible by } n \in \mathbb{N}$$

$\mathcal{L}(M)$ set of all lower states of M

$\mathcal{K}_{\mathcal{S}(M)}$ family of all nonempty compact convex sets in $\mathcal{S}(M)$

Theorem (Fedel, Keimel, Montagna and Roth, 2013)

Let M be a 2-divisible MV-algebra. There is a bijection between $\mathcal{L}(M)$ and $\mathcal{K}_{\mathcal{S}(M)}$ given by

$$t \in \mathcal{L}(M) \quad \mapsto \quad \mathcal{C}(t) := \{s \in \mathcal{S}(M) \mid s(a) \geq t(a) \ \ \forall a \in M\}$$

and

$$\mathcal{C} \in \mathcal{K}_{\mathcal{S}(M)} \quad \mapsto \quad t_{\mathcal{C}}(a) := \min\{s(a) \mid s \in \mathcal{C}\}$$

lower states of M ⟷ lower states of G_2

$\overline{t}(a) := 2^n \cdot t\left(\frac{1}{2^n}a\right)$

$t'(b) := \overline{t}(b + m\mathbf{1}) - m$

lower states of M ⟷ compact convex subsets of $\mathcal{S}(M)$

lower states of G_2 ⟷ norm. superlinear functionals on $C(X_M)$ (Stone Weierstrass)

compact convex subsets of $\mathcal{S}(M)$ ⟷ norm. superlinear functionals on $C(X_M)$ (min / $\mathcal{C}$)

Proof by embedding $M \hookrightarrow G_2 \hookrightarrow C(X_M)$

- ✓ characterization of lower states by duality with sets of states
- ✓ coherence criterion for non-reversible bookmaking

- ⋆ works only for 2-divisible MV-algebras
- ⋆ avoids the use of good sequences
- ⋆ important special cases are not discussed...

COMONOTONICALLY ADDITIVE LOWER STATES

Hardy, Littlewood, Pólya (1934) Schmeidler (1986)

Definition

Let M be a semisimple MV-algebra. Elements $a, b \in M$ are comonotonic (we write $a \approx b$) if for all valuations h, h',

$$h(a) \geq h'(a), \;\; h(b) \geq h'(b) \qquad \text{or} \qquad h(a) \leq h'(a), \;\; h(b) \leq h'(b).$$

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Example

- $M =$ boolean algebra, every chain is a comonotonic class
- any elements $a, b \in M$ with $a \oplus b = a$ are comonotonic
- $M = [0,1]^n$, $S_\pi := \{\mathbf{x} \in [0,1]^n \mid x_{\pi(1)} \leq \cdots \leq x_{\pi(n)}\}$, where $\pi \in \Pi_n$

COMONOTONICALLY ADDITIVE FUNCTIONALS

Definition

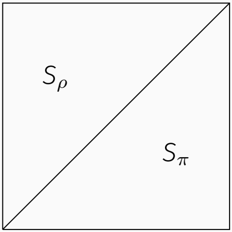
Let M be an MV-algebra. A functional $t\colon M \to [0,1]$ is comonotonically additive if for every $a, b \in M$ such that $a \odot b = \mathbf{0}$ and $a \approx b$ we have

$$t(a \oplus b) = t(a) + t(b).$$

- meaningful only when M is not a boolean algebra
- we are interested in comonotonically additive lower states

Let M be the finite direct product of standard MV-algebras $[0,1]$.
Then the maximal comonotonicity classes are the simplices

$$S_\pi = \{\mathbf{x} \in [0,1]^n \mid x_{\pi(1)} \leq \cdots \leq x_{\pi(n)}\}, \qquad \pi \in \Pi_n$$



$n = 2$

COMONOTONICALLY ADDITIVE LOWER STATES OF $[0,1]^n$

Proposition

Comonotonically additive lower states of $[0,1]^n$ are exactly the concave Lovász functions $t\colon [0,1]^n \to [0,1]$, that is, each such t is

- continuous PL over each S_π, $\pi \in \Pi_n$
- concave
- $t(\mathbf{1}) = 1$ and $t(\mathbf{0}) = 0$

Theorem

There is a bijection between

1. Comonotonically additive lower states of $[0,1]^n$
2. Generalized permutohedra in the standard n-simplex

1. Characterize the comonotonicity classes in an MV-algebra
2. Relax divisibility of M, but require comonotonicity of t
3. Is there a bookmaking scheme attached to comonotonically additive lower states?