Monadic Fragments of Modal Predicate Logics

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Motivation & History

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What about intuitionistic predicate logic (QIPC)?

Prior 1955

Introduces modal intuitionistic propositional calculus MIPC as an intuitionistic analog of S5.

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What about modal predicate logics?

Goal

Axiomatize monadic fragments of modal predicate logics using products and relativized products of Kripke frames.

Monadic Modal Logics

Language $\mathscr{L}_{MM}$

- classical modal language $\mathscr{L}_M$
- the monadic operator $\forall$
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- substitution
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Least set of formulas of $\mathscr{L}_{MM}$ that contains:

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Unlike propositional modal logics, many modal predicate logics are not Kripke complete.

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- $T(\forall\varphi) = \forall x T(\varphi)$

Galois Connection

For a mm-logic $\mathsf{L} \supseteq \mathsf{mK}$, define a modal predicate logic:

$$\Phi(\mathsf{L}) = \mathsf{QK} + \{T(\varphi) : \mathsf{L} \vdash \varphi\}$$

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Lemma

For $\mathsf{L} \supseteq \mathsf{mK}$ and $\mathsf{M} \supseteq \mathsf{QK}$

1. Φ and Ψ form a Galois connection, that is $\Phi(\mathsf{L}) \subseteq \mathsf{M}$ iff $\mathsf{L} \subseteq \Psi(\mathsf{M})$.
2. $\Psi(\Phi(\mathsf{L})) \supseteq \mathsf{L}$ with equality iff $\mathsf{L} = \Psi(\mathsf{M})$ for some $\mathsf{M} \supseteq \mathsf{QK}$.
3. $\mathsf{M} \supseteq \Phi(\Psi(\mathsf{M}))$ with equality iff $\mathsf{M} = \Phi(\mathsf{L})$ for some $\mathsf{L} \supseteq \mathsf{mK}$.

Monadic Fragment

Definition

We call $\mathsf{L} \supseteq \mathsf{mK}$ the monadic fragment of a modal predicate logic $\mathsf{M} \supseteq \mathsf{QK}$ if

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Goal

Develop a correspondence between models of mm-logics and models of modal predicate logics, in order to obtain results similar to those of Ono and Suzuki.

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mK-frame

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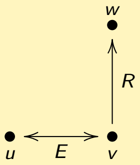

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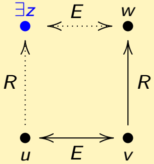

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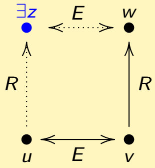

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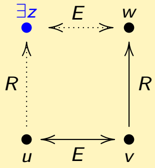

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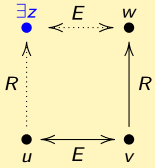

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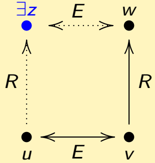

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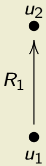
⚠ To translate between mK-frames and predicate Kripke frames we need to work with a much smaller class of mK-frames, arising from product frames.

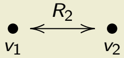
Products

Product Frame

- $\mathfrak{F}_1 = \langle W_1, R_1 \rangle \quad \times \quad \mathfrak{F}_2 = \langle W_2, R_2 \rangle$

Example





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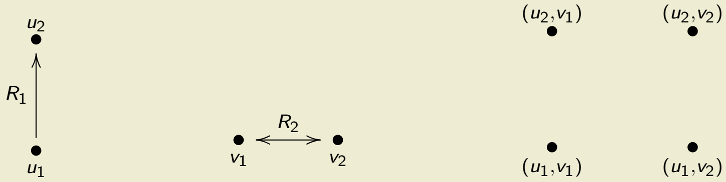

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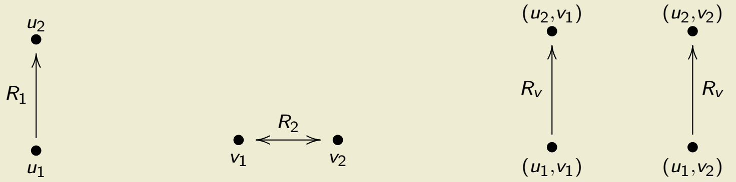

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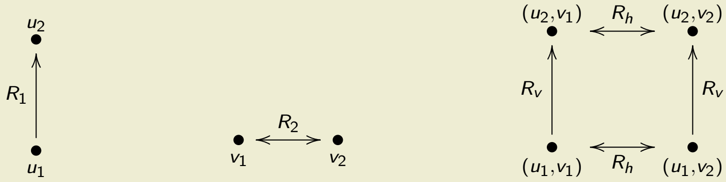

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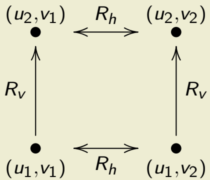

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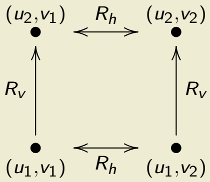

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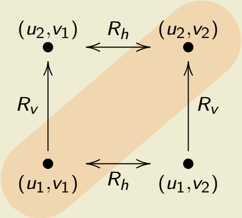

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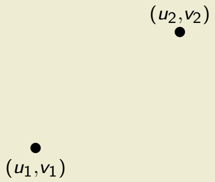

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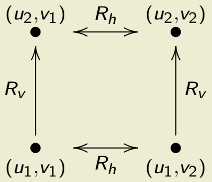

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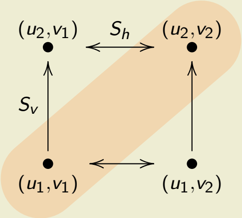

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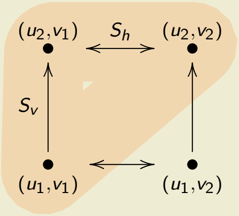

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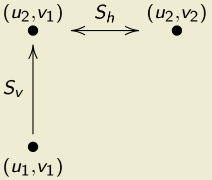
- $W \subseteq W_1 \times W_2$
- S_i is the restriction of R_i to W for $i = h, v$

Expanding Relativized Product (ERP)

- RP of $\mathfrak{F}_1 \times \mathfrak{F}_2$
- for $(u_i, v_j) \in W$ and $u_k \in W_1$, if $u_i R_1 u_k$ then $(u_k, v_j) \in W$

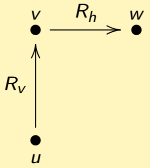
Example



$$\underbrace{\mathfrak{F} = \langle W, S_v, S_h \rangle}_{\text{ERP}}$$

Properties of Product Frames

Left Commutativity (com^l)



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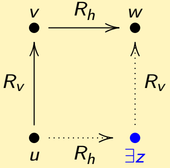

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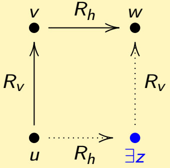
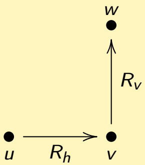


Right Commutativity (com^r)



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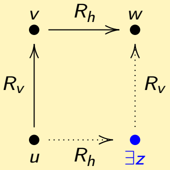

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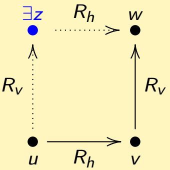

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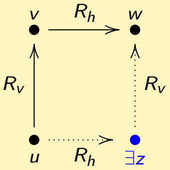

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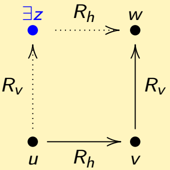
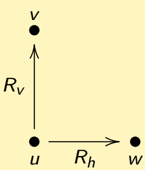


Church-Rosser Property (chr)



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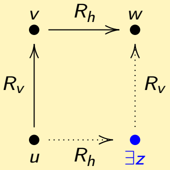

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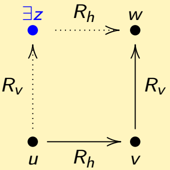

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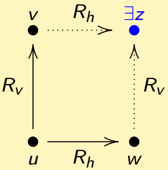

Properties of ERP Frames

In our ERP frames $\mathfrak{F} = \langle W, S_v, S_h \rangle$, we take subframes of $\mathfrak{F}_1 \times \mathfrak{F}_2$ where $R_2 = W_2 \times W_2$ (modeling our S5 modality $\forall$),

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Some Notes

- chr is automatic when one of $\mathfrak{F}_1$ or $\mathfrak{F}_2$ is an S5-frame
- full commutativity (*com*) $\Leftrightarrow$ Barcan formula (full product frames)
- We lose half of commutativity when restricted to ERP frames

ERP Frames $\rightarrow$ Predicate Frames

Constant Domains

$$\underset{\text{ERP frame}}{\mathfrak{F} = \langle W, R, E \rangle} \qquad \longrightarrow \qquad \underset{\text{predicate Kripke frame}}{\mathfrak{F}^\dagger = \langle W^\dagger, R^\dagger, D \rangle}$$

ERP Frames → Predicate Frames

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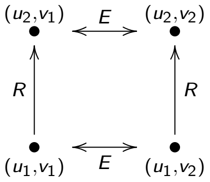

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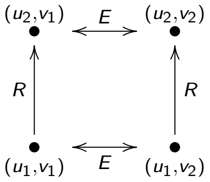

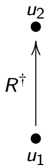

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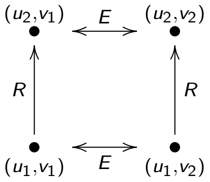

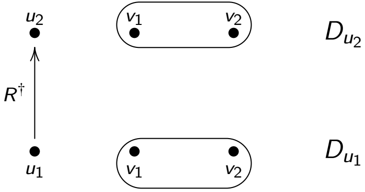

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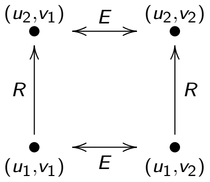

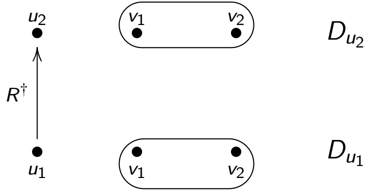

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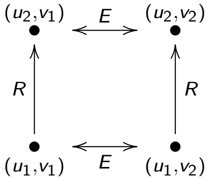

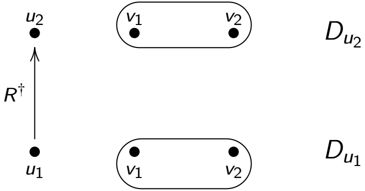

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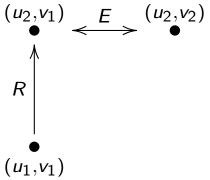

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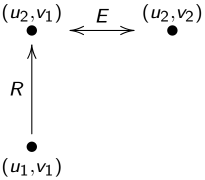

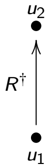

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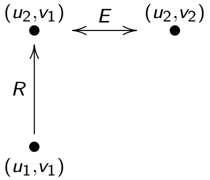

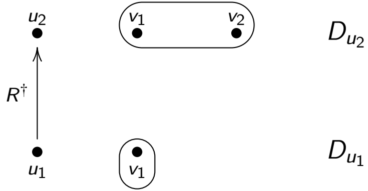

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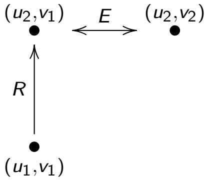

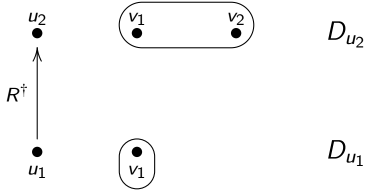

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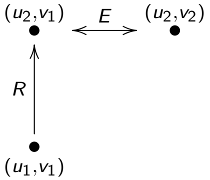

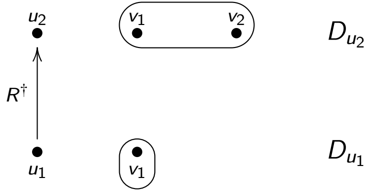

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ERP frame — predicate Kripke frame

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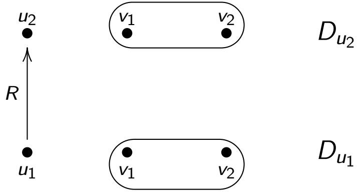

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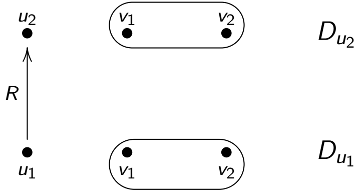

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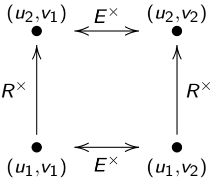

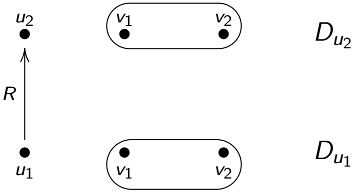

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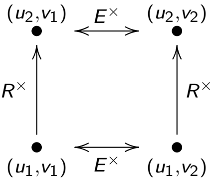

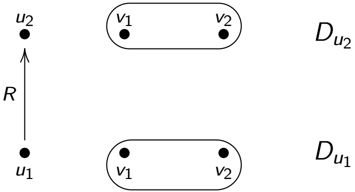

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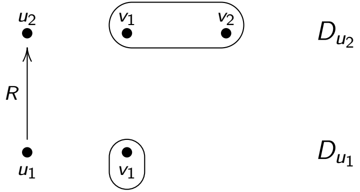

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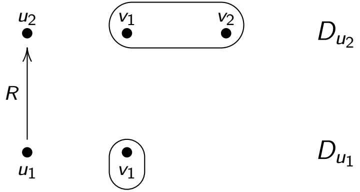

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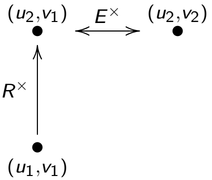

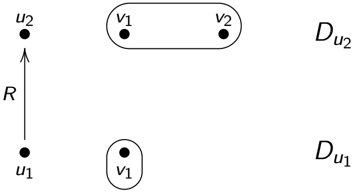

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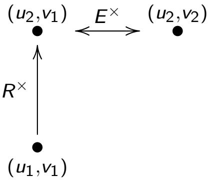

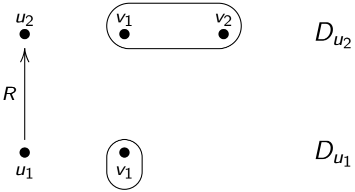

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Translating Frames

Theorem

1. *If $\mathfrak{F}$ is an ERP frame and $\varphi \in$ **Form**$(\mathscr{L}_{MM})$, then*

$$(\mathfrak{F}, (u, v)) \models \varphi \quad \textit{iff} \quad (\mathfrak{F}^{\dagger}, u) \models (T(\varphi))^{v}_{x}.$$

2. *If $\mathfrak{F}$ is a predicate Kripke frame and $\varphi \in$ **Form**$(\mathscr{L}_{MM})$, then*

$$(\mathfrak{F}, u) \models (T(\varphi))^{v}_{x} \quad \textit{iff} \quad (\mathfrak{F}^{\times}, (u, v)) \models \varphi.$$

Translating Frames

Theorem

1. *If $\mathfrak{F}$ is an ERP frame and $\varphi \in \mathbf{Form}(\mathscr{L}_{MM})$, then*

$$(\mathfrak{F},(u,v)) \vDash \varphi \quad \textit{iff} \quad (\mathfrak{F}^{\dagger},u) \vDash (T(\varphi))^{v}_{x}.$$

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Note

This will ultimately allow us to generalize Ono & Suzuki's results to monadic modal logics.

Completeness

Theorem (Gabbay, Kurucz, Wolter, Zakharyaschev, 2003)

1. mK *is complete with respect to the class of all ERP frames, and for* $\mathsf{L} \in \{\mathsf{K4}, \mathsf{S4}, \mathsf{S5}\}$, mL *is complete with respect to the class of all ERP frames for which* R *is either transitive (*K4*), a quasi-order (*S4*), or an equivalence relation (*S5*).*
2. bK *is complete with respect to the class of all product frames, and for* $\mathsf{L} \in \{\mathsf{K4}, \mathsf{S4}\}$, bL *is complete with respect to the class of all product frames for which* R *is either transitive (*K4*) or a quasi-order (*S4*).*

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Just as Ono & Suzuki adjusted the well-known Henkin construction for intuitionistic modal logics, we can adjust similarly for mm-logics for a simpler proof of the above theorem. ▸▸ We don't have time for this!

Modified Henkin Construction

Start as usual...

1. $\mathsf{mK} \nvdash \varphi$, set $\Gamma_{00} = \{\neg\varphi\}$
2. Enumerate all formulas of $\mathscr{L}_0 = \mathscr{L}_{MM}$ as $\psi_1, \psi_2, \ldots$
3. $\Gamma_{0i+1} =$
$$\begin{cases} \Gamma_{0i} \cup \{\psi_{i+1}\} & \text{if } \Gamma_{0i} \cup \{\psi_{i+1}\} \text{ is consistent} \\ \Gamma_{0i} \cup \{\neg\psi_{i+1}\} & \text{if } \Gamma_{0i} \cup \{\neg\psi_{i+1}\} \text{ is consistent} \end{cases}$$
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Add "witnesses"

5. Let v_{ij} $(i, j = 1, 2, 3, \ldots)$ be new variables not occurring in $\mathscr{L}_0$ and let $V_1 = \emptyset$
6. Enumerate all formulas of Γ_0 of the form $\exists\psi$ as $\exists\chi_1, \exists\chi_2, \ldots$
7. Add the formula χ_j to Γ_0 and add the new variable v_{1j} to V_1
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Construct the model

$\mathfrak{M} = \langle W, R, E, \mathfrak{V} \rangle$

- Let $W = \{(\Gamma, v) : v \in V_\Gamma\}$
- $(\Gamma, v)R(\Delta, u)$ iff $\Box\psi \in \Gamma \Rightarrow \psi \in \Delta$ for all formulas ψ and $v = u$
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When working with bK, we simply take V to be the collection of all variables and $W = \{(\Gamma, v) : v \in V\}$.

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Theorem

Let $\mathsf{L} \supseteq \mathsf{mK}$ *be a mm-logic complete with respect to a class* $\{\mathfrak{F}_i\}_{i \in I}$ *of ERP frames. If* $\mathsf{M} \supseteq \mathsf{QK}$ *is sound with respect to* $\{\mathfrak{F}_i^{\dagger}\}_{i \in I}$*, then* $\langle \mathsf{L}; \mathsf{M} \rangle$*.*

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For $\mathsf{L} \in \{\mathsf{K}, \mathsf{K4}, \mathsf{S4}, \mathsf{S5}\}$,

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  - The proof required $\mathrm{mS4} \vdash \varphi \Rightarrow \mathrm{QS4} \vdash T(\varphi)$, but the other implication was left open.
  - Now we can see that the other implication holds as well and give a simplified version of her proof that $\mathrm{MIPC} \vdash \varphi$ iff $\mathrm{mS4} \vdash \varphi^t$.

Thank You!