Problems on the frontier of commutator theory

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TACL
Ischia, Italy
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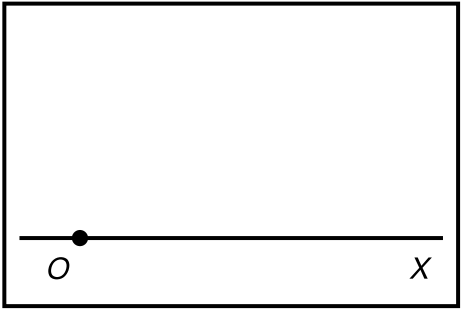

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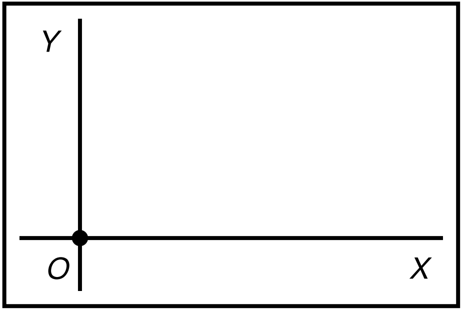

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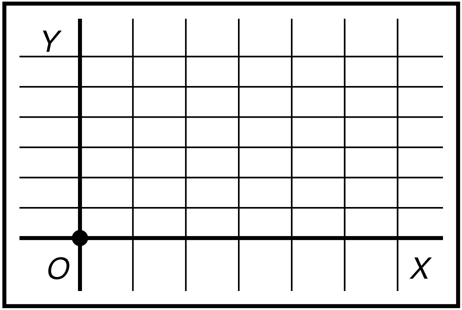

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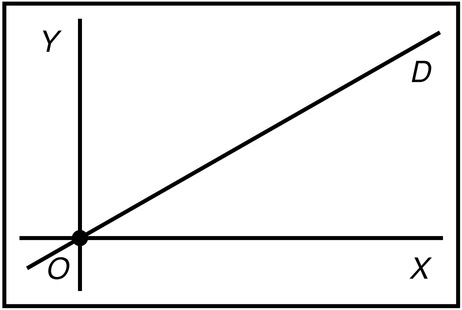

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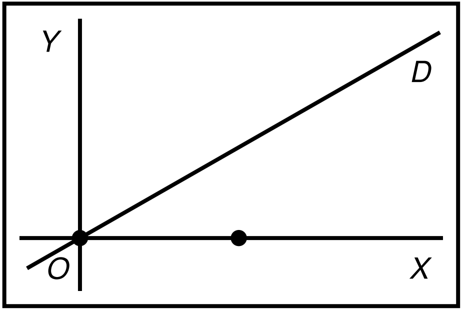

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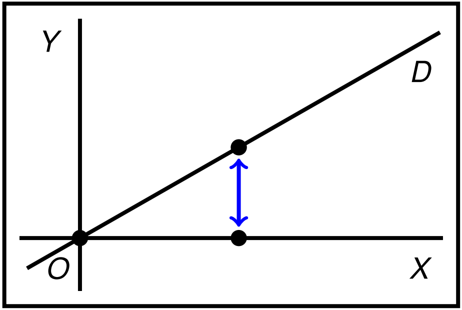

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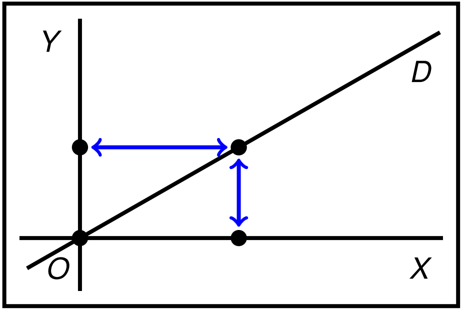

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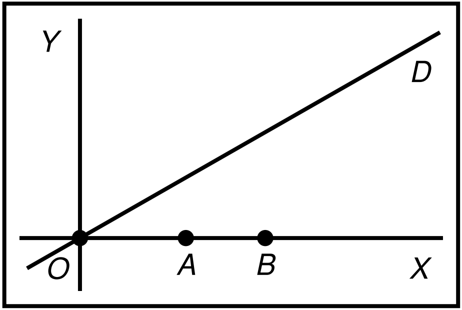

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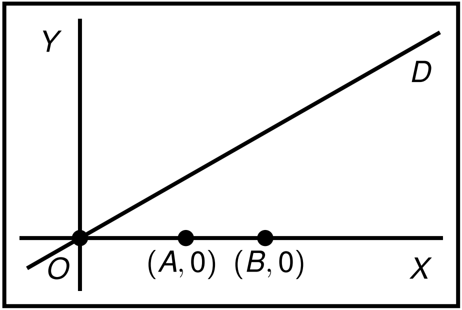

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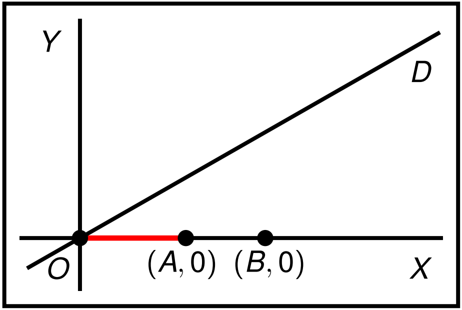

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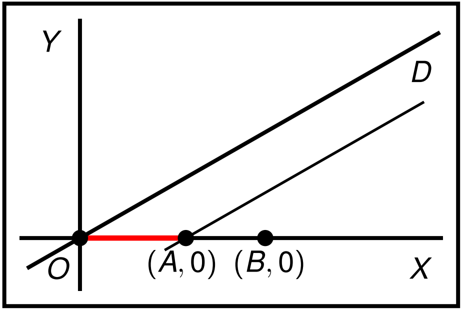

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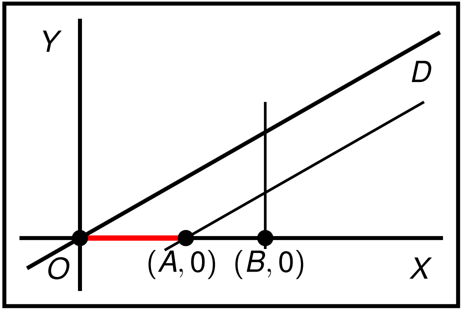

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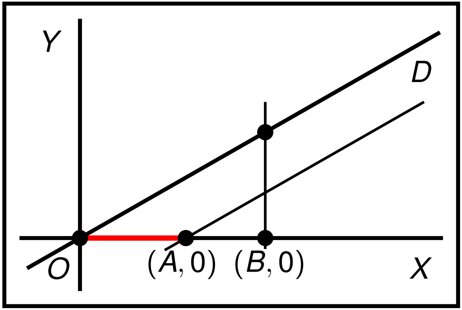

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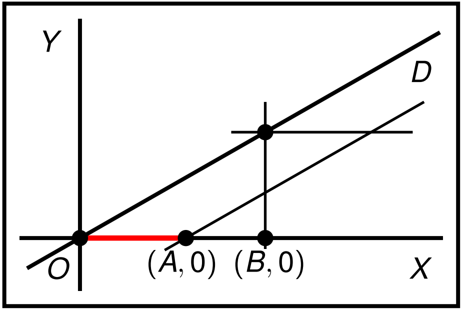

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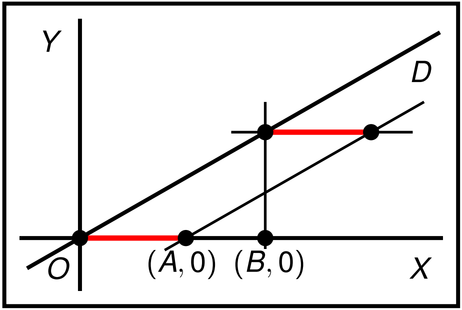

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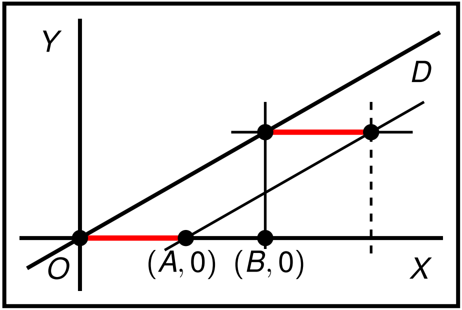

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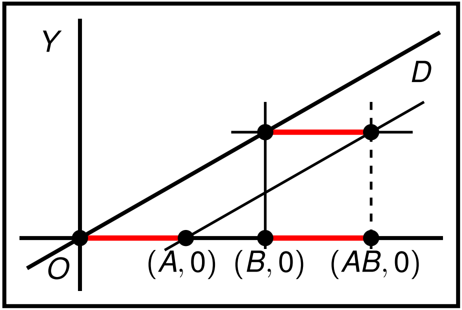

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3. Eliminating the choice of a point, the information can really be encoded in a "ternary loop" structure on a line, where $xy = m(x, 0, y), x/y = r(x, 0, y)$ and $x\backslash y = \ell(x, 0, y)$. This $m(x, y, z)$ satisfies $m(x, x, z) = z = m(z, x, x)$.

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4. Any algebra A has the property that $A \times A$ has congruences defining "vertical" and "horizontal" line families. If it has a good third congruence, then A will have a compatible ternary loop/Maltsev structure.

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4. Any algebra A has the property that $A \times A$ has congruences defining "vertical" and "horizontal" line families. If it has a good third congruence, then A will have a compatible ternary loop/Maltsev structure.
5. If A has "strong enough" operations, the only possible compatible Maltsev operation on A is $x - y + z$ with respect to some abelian group structure on A.

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α centralizes β if $\Delta_{\alpha,\beta} \cap \pi_1 = 0 = \Delta_{\alpha,\beta} \cap \pi_2$.
$[\alpha, \beta]$ is the least $\gamma \in \mathrm{Con}(A)$ such that $\Delta_{\alpha,\beta} \cap \gamma_1 = \Delta_{\alpha,\beta} \cap \gamma_2$.

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2. It also generalizes ideal product and Lie bracket.
3. It is reasonably easy to calculate with this commutator.

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Here is the "$+\epsilon$":
Given an R-module M and a submodule $U \leq R \times M$, equip the set M with all operations of the form

$$r_1(x_1) + r_2(x_2) + \cdots + r_n(x_n) + m$$

where $(1 - \sum r_i, m) \in U$.

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$(C; \{rx + (1-r)y \mid 0 < r < 1\})$, where C is a convex subset of $\mathbb{R}^n$, is a subalgebra of a reduct of $(\mathbb{R}^n; +, -, 0, \{rx \mid r \in \mathbb{R}\})$.

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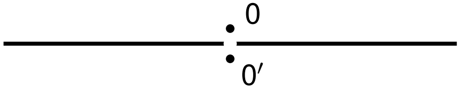
Still worse: the doubly pointed line is abelian but not quasiaffine.



$(\mathbb{R} \cup \{0'\}; +, +')$

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2. *(Kearnes & Szendrei, 1998) If the congruence lattices of algebras in $\mathcal{V}$ satisfy a nontrivial law in $\{\vee, \wedge\}$, then abelian algebras in $\mathcal{V}$ are affine.*
3. *(K & S) If the congruence lattices of algebras in $\mathcal{V}$ satisfy a nontrivial law in $\{\circ, \wedge\}$, then abelian algebras in $\mathcal{V}$ are quasiaffine but need not be affine.*

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3. *(K & S) If the congruence lattices of algebras in $\mathcal{V}$ satisfy a nontrivial law in $\{\circ, \wedge\}$, then abelian algebras in $\mathcal{V}$ are quasiaffine but need not be affine.*
4. *If $\mathcal{Q}$ is a relatively congruence modular quasivariety, then abelian algebras in $\mathcal{V}$ are quasiaffine but need not be affine.*

Applications: finite basis theorems

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2. *(Kearnes and Willard) There exists a finitely generated abelian variety with no finite equational basis.*
3. *(K & W) Every finitely generated abelian variety is contained in a finitely generated abelian variety that has a finite equational basis.*

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1. Is it decidable if a finite abelian algebra has a finite equational basis?
2. Does every finite quasiaffine algebra have a finite equational basis?
3. (Pigozzi) Does every finitely generated, relatively congruence modular, abelian quasivariety have a finite quasiequational basis?

Another specific question

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1. Is there a good description of abelian algebras in more general “modular categories”?

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Problem: We don't know how to control the sizes of the rings and modules that arise this way without further assumptions.

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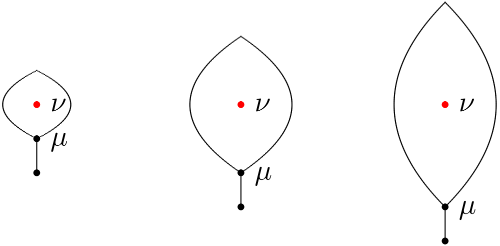
An example: Ross Willard and I showed that if all subdirectly irreducible algebras are finite in a congruence modular variety with finitely many basic operations, then there are only finitely many of them.

Sketch of idea

SI's: A_1, A_2, A_3, $|A| \nearrow \omega$

Con's:



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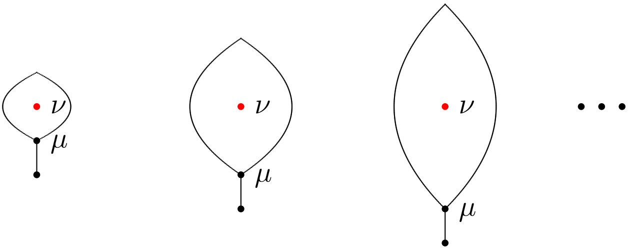

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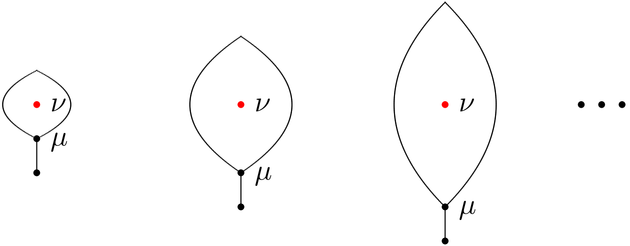

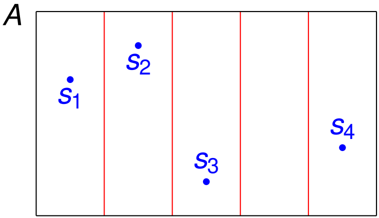

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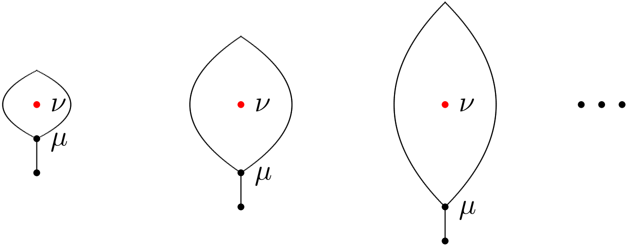
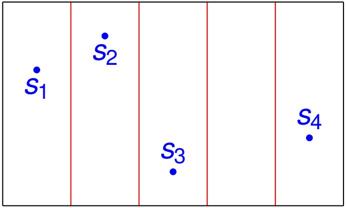


A



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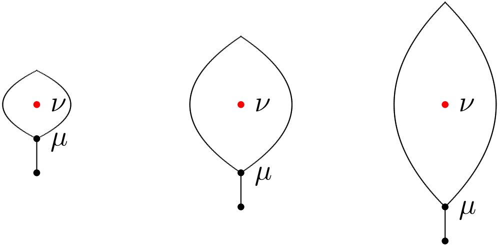

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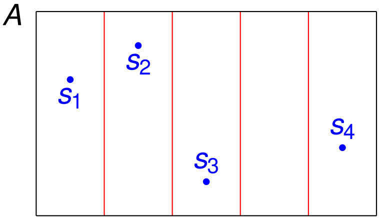

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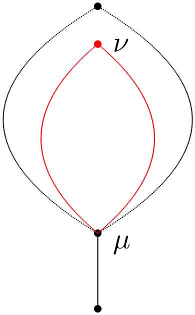
Cases: $|S| \nearrow \omega$ or not

Sketch of idea

$\mathrm{Con}(A)$, A SI



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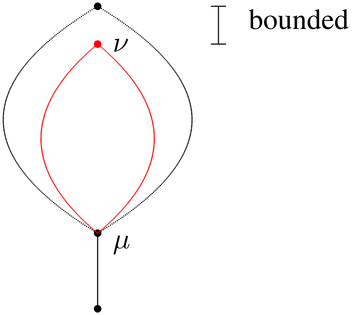

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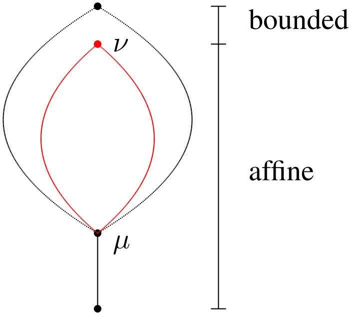

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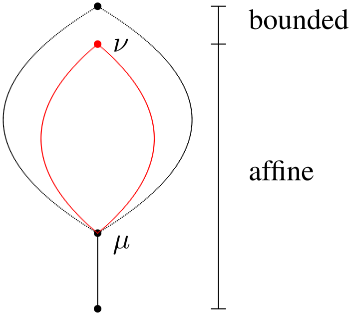

Want a ring, built from $A/(0 : \nu)$, acting on different algebras. Need a uniform construction.

Specific problems

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1. Let $\mathcal{V}$ be a variety whose abelian congruences are affine. Is it true that the polynomial structure on an abelian congruence θ is determined by the structure of $A/(0 : \theta)$?
2. Systematize the construction of rings acting on abelian congruences.
3. Can anything like this be done for relatively congruence modular quasivarieties whose abelian congruences are not affine?

3. Is there an easy way to decide if $[\alpha, \beta] = 0$?

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Model: For modular varieties or quasivarieties we know that the relation $[\mathrm{Cg}(a, b), \mathrm{Cg}(c, d)] = 0$ is equational, meaning that it is defined by a conjunction of sentences of the form

$$\forall \bar{e}(s(a, b, c, d, \bar{e}) = t(a, b, c, d, \bar{e})).$$

In the case of varieties, Ralph McKenzie found the equations explicitly. The equations have been used to prove finite basis theorems and theorems about the distribution of subdirectly irreducible algebras.

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1. What are the equations for modular quasivarieties?
2. What are the formulas for nonmodular varieties whose abelian congruences are affine? These formulas would help in defining centralizers, $(0 : \mu)$, and in defining nilpotent and solvable radicals.