Category theory and Homotopy Type theory

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References

Main authors:

- Per Martin-Löf (1971, 1975, 1984)
- Martin Hofmann, Thomas Streicher (1995)
- Steven Awodey, Michael Warren (2006)
- Vladimir Voevodsky (2006)

General references:

HTT Book:

http://homotopytypetheory.org/book/

Plan of my talk

- A sketch of type theory
- Models
- The notion of tribe
- The category of tribes

A sketch of type theory

- Types, elements, judgments and contexts
- Judgmental equality
- Substitution rules
- Σ-formation and introduction rules
- Π-formation and introduction rules
- Propositional equality
- Universes
- Univalence

Types and judgments

The basic notion is that of a **type** ($\simeq$ set).

The assertion that A is a type is formally expressed by writing

$$\vdash A : \mathit{Type}$$

This expression is an instance of what is called a **judgment**.

For example, the judgment

$$\vdash \mathbb{N} : \mathit{Type}$$

asserts that the set $\mathbb{N}$ of natural numbers is a type.

Elements, terms

The assertion that x is an **element** of type A is formally expressed by the judgment

$$\vdash x : A$$

For example, the judgment

$$\vdash 0 : \mathbb{N}$$

asserts that 0 is a natural number.

An element $x : A$ is often called a **term**.

There are terms forming operations

$$t ::= x \mid \lambda x.t \mid t(t') \mid c \mid f$$

Judgmental equality

In Martin-Löf type theory, two objects are **intentionally equal** if they have the **same normal form**.

The assertion that two elements x and y of type A are intentionally equal is written as a judgment:

$$\vdash x \equiv y : A$$

Also the assertion that two types A and B are intentionally equal,

$$\vdash A \equiv B$$

Dependant types and contexts

A type $B(x)$ may depend on a parameter x ranging in a type A.

$$x : A \vdash B(x) : \mathit{Type}$$

The expression $x : A$ on the left of the symbol $\vdash$ is the **context** of the judgment.

An element of type $B(x)$ may depend on x:

$$x : A \vdash t(x) : B(x)$$

Contexts may be concatenated:

$$y : B(x),\ x : A \vdash C(x, y) : \mathit{Type}$$

Substitution rules

There are *substitution rules* for types and terms:

$$\frac{x : A \vdash f(x) : B, \qquad y : B \vdash E(y) : \mathit{Type}}{x : A \vdash E(f(x)) : \mathit{Type}}$$

$$\frac{x : A \vdash f(x) : B, \qquad y : B \vdash s(y) : E(y)}{x : A \vdash s(f(x)) : E(f(x))}$$

Σ-formation rules

There is a *formation rule* for the sum (= disjoint union) of a dependant type $E(x)$ in context $x : A$.

$$\frac{x : A \vdash E(x) : \mathit{Type}}{\vdash \sum_{x:A} E(x) : \mathit{Type}}$$

There is an *introduction rule* for pairs:

$$\frac{\vdash a : A, \qquad \vdash b : E(a)}{\vdash (a, b) : \sum_{x:A} E\langle x \rangle}$$

Π-formation rules

There is a *formation rule* for the product of a dependant type $E(x)$ in context $x : A$.

$$\frac{x : A \vdash E(x) : Type}{\vdash \prod_{x:A} E(x) : Type}$$

There is an *introduction rule* for λ-term and an *elimination rule*:

$$\frac{x : A \vdash t : E(x)}{\vdash \lambda x.t : \prod_{x:A} E(x)} \qquad \frac{\vdash f : \prod_{x:A} E(x), \qquad \vdash a : A}{\vdash f(a) : E(a)}$$

As usual, the term $\lambda x.t(x)$ stands for the map $x \mapsto t(x)$.

Computation and uniqueness rules

There is a computation rule:

$$\frac{\vdash \lambda x.t(x) : \prod_{x:A} E(x), \qquad \vdash a : A}{\vdash (\lambda x.t(x))(a) \equiv t(a)}$$

and a uniqueness rule:

$$\frac{\vdash f : \prod_{x:A} E(x)}{\vdash f \equiv \lambda x.f(x)}$$

Equality type

There is a type constructor which associates to a type A another type Eq_A, called the **equality type** of A,

$$\frac{\vdash \ A : Type}{x, y : A \ \vdash Eq_A(x, y) : Type}$$

A term $p : Eq_A(x, y)$ is a *proof* that $x \simeq y$ (propositional equality).

The axiom that $x \simeq x$ is given by a term $r(x)$ called the **reflexivity term**:

$$\frac{A : Type}{x : A \ \vdash r(x) : Eq_A(x, x)}$$

The J-operation

There is a term constructor J which associates to a dependant type

$$z : Eq_A(x, y),\ x, y : A \vdash E(z) : Type$$

together with a term $x : A \vdash t : E(r(x))$, another term

$$z : Eq_A(x, y),\ x, y : A \ \vdash J(t)(z) : E(z).$$

There is also a *computation rule*:

$$x : A \vdash J(t)(r(x)) \equiv t : E(r(x))$$

Universes

We postulate an infinite sequences of universes

$$\vdash \mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 : \cdots$$

with the axioms:

(1) Every type A is a term in some universe $\mathcal{U}_i$.

$$\vdash A : \mathcal{U}_i$$

(2) Every term in $\mathcal{U}_i$ is a term in $\mathcal{U}_{i+1}$.

$$\frac{A : \mathcal{U}_i}{A : \mathcal{U}_{i+1}}$$

The groupoid model

Hofmann and Streicher:

Type theory has a model in groupoids.

- types $\rightsquigarrow$ groupoids;
- terms $\rightsquigarrow$ objects of a groupoid;
- dependant types in context A $\rightsquigarrow$ fibrations $E \twoheadrightarrow A$;
- proofs that $a \simeq b$ $\rightsquigarrow$ isomorphisms $a \to b$.

The simplicial set model

Awodey, Warren, Veovodsky :

Type theory has a model in simplicial sets

- types $\rightsquigarrow$ Kan complexes;
- terms $\rightsquigarrow$ vertices of a Kan complex;
- dependant types in context A $\rightsquigarrow$ Kan fibrations $E \twoheadrightarrow A$;
- proofs that $a \simeq b$ $\rightsquigarrow$ paths $a \to b$.

Fibrations and dependant types

The **fiber** $E(x)$ of a fibration $p : E \to A$ at a point $x \in A$ is defined by the pullback square

$$\begin{array}{ccc} E(x) & \longrightarrow & E \\ \downarrow & & \downarrow\!\!\downarrow p \\ \star & \xrightarrow{\;x\;} & A. \end{array}$$

A fibration $p : E \twoheadrightarrow A$ can be regarded as a **family** $(E(x) : x \in A)$ of objects parametrized by a variable element $x \in A$.

A fibration $p : E \twoheadrightarrow A$ is a **dependant type** in context A.

Equivalences

For any two types A and B, there is a type $Equiv(A, B)$ whose element are the equivalences $A \cong B$.

An **equivalence** $w : A \cong B$ is a quintuple $w \equiv (f, g_1, g_2, h_1, h_2)$, where

$$f : A \to B \quad \text{and} \quad g_1, g_2 : B \to A$$

are maps and

$$h_1 : g_1 \circ f \simeq id_A \quad \text{and} \quad h_2 : f \circ g_2 \simeq id_B$$

are terms.

Univalence

For any type $B(x)$ which depends on $x : A$ there is a canonical map

$$\alpha : Eq_A(x, y) \to Equiv(B(x), B(y))$$

which depends on $x, y : A$.

Univalence axiom (Voevodsky): If $\mathcal{U}$ is an universe, then the map

$$\alpha : Eq_{\mathcal{U}}(A, B) \to Equiv(A, B)$$

is an equivalence for every $A, B : \mathcal{U}$.

Results and problems

Results:

- Shulman: $\pi_1(S^1) = \mathbb{Z}$
- Licata: $\pi_n(S^n) = \mathbb{Z}$ for $n > 0$, $\pi_k(S^n) = 0$ for $k < n$,
- Brunerie: $\pi_3(S^2) = \mathbb{Z}$
- Lumsdane, Finster, Licata: Freudenthal suspension theorem.

Problems:

- $\pi_4(S^3) = \mathbb{Z}/2$?
- No good notion of (internal) simplicial object
- No notion of $(\infty, 1)$-categories.

Axiomatic Homotopy Theory

J.H.C. Whitehead (1950):

The ultimate aim of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that analytic is equivalent to pure projective geometry.

Examples of axiomatic systems in homotopy theory:

- Triangulated categories (Verdier 1963);
- Homotopical algebra (Quillen 1967);
- Fibration categories (Brown 1973);
- Homotopy theories (Heller 1988)
- Theory of derivators (Grothendieck 198?)
- Homotopy type theory

Fibration structure

Let $\mathcal{C}$ be a category with terminal object $\star$.

Definition

A **fibration structure** on $\mathcal{C}$ is a class of maps $\mathcal{F} \subseteq \mathcal{C}$ called **fibrations** and denoted $A \twoheadrightarrow B$, satisfying the following conditions:

- Every isomorphism is a fibration;
- The composite of two fibrations is a fibration;
- the base change of fibration along any map exists and is a fibration;

$$\begin{array}{ccc} A \times_B E & \longrightarrow & E \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow{\ u\ } & B \end{array}$$

- the map $X \to \star$ is a fibration for every object $X \in \mathcal{C}$.

Anodyne maps

Let $\mathcal{C}$ be a category equipped with a fibration structure.

Definition

A map $u : A \to B$ in $\mathcal{C}$ is said to be is **anodyne** if it has the left lifting property with respect to every fibration $f : X \twoheadrightarrow Y$.

This means that every commutative square

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle u}\downarrow & & \downarrow\!\!\downarrow{\scriptstyle f} \\ B & \xrightarrow{b} & Y \end{array}$$

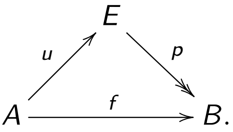
has a diagonal filler $d : B \to X$ ($du = a$ and $fd = b$).

Tribes

Definition

A **tribe** $\mathcal{C}$ is a category equipped with a fibration structure such that:

- every map $f : A \to B$ admits a factorization $f = pu$ with u anodyne and p a fibration.

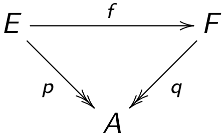


- the base change of an anodyne map along a fibration is anodyne.

The local tribe $\mathcal{C}(A)$

Let A be an object of a tribe $\mathcal{C}$.

The **local tribe** $\mathcal{C}(A)$ is defined to be the full sub-category of $\mathcal{C}/A$ whose objects are the fibrations $p : E \twoheadrightarrow A$ with codomain A.



A map $f : (E, p) \to (F, q)$ in $\mathcal{C}(A)$ is a fibration if the map $f : E \to F$ is a fibration in $\mathcal{C}$.

An object of $\mathcal{C}(A)$ is a **dependent type** in **context** A.

Homomorphism of tribes

If $\mathcal{C}$ and $\mathcal{D}$ are tribes.

A functor $F : \mathcal{C} \to \mathcal{D}$ is a **homomorphism** if

- it takes fibrations to fibrations;
- it takes anodyne maps to anodyne maps;
- it preserves base changes of fibrations;
- it preserves terminal objects.

For example, if $u : A \to B$ is a map in a tribe $\mathcal{C}$, then the base change functor $u^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is a homomorphism of tribes.

Remark: The category of tribes is a 2-category, where a 1-cell is a homomorphism and 2-cell is a natural transformation.

Product along a fibration

A fibration $f : A \to B$ induces a base change functor

$$f^\star : \mathcal{C}/B \to \mathcal{C}/A$$

The **product along** $f : A \to B$ of an object $E = (E, p) \in \mathcal{C}/A$ is an object $\Pi_f(E) \in \mathcal{C}/B$ equipped with a map

$$\epsilon : f^\star(\Pi_f(E)) \to E$$

which is couniversal with respect to the functor $f^\star$.

For every $y : B$ we have

$$\Pi_f(E)(y) = \prod_{f(x)=y} E(x)$$

π-tribes

Definition

We say that a tribe $\mathcal{C}$ is a π-**tribe**, if for every fibration $f : A \to B$

- the product $\Pi_f(E)$ exists for every $E \in \mathcal{C}(A)$ and the structure map $\Pi_f(E) \to B$ is a fibration.
- the functor $\Pi_f : \mathcal{C}(A) \to \mathcal{C}(B)$ preserves anodyne maps.

If $\mathcal{C}$ is a π-tribe, then so is the tribe $\mathcal{C}(A)$ for every object $A \in \mathcal{C}$.

Path objects in a Quillen model category

If $\mathcal{M}$ is a Quillen model category.

A **path object** for $A \in \mathcal{M}$ is obtained by factoring the diagonal $\Delta : A \to A \times A$ as an anodyne map $r : A \to PA$ followed by a fibration $(s,t) : PA \twoheadrightarrow A \times A$,

$$\begin{array}{ccc} & PA & \\ {\scriptstyle r}\nearrow & & \searrow{\scriptstyle (s,t)} \\ A & \xrightarrow{\quad\Delta\quad} & A \times A. \end{array}$$

The identity type is a path object

Awodey-Warren:

The identity type $Eq_A \to A \times A$ is a path object for A.

The J-rule implies that the reflexivity term $r : A \to Eq_A$ is anodyne.

Thus, if $p : E \twoheadrightarrow Eq_A$ is a fibration, then every commutative square

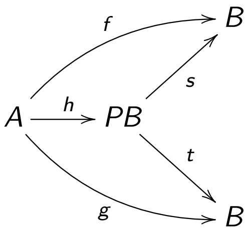
$$\begin{array}{ccc} A & \xrightarrow{\ t\ } & E \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} \\ Eq_A & = & Eq_A \end{array}$$

has a diagonal filler $d = J(t)$,

Homotopic maps

Let $\mathcal{C}$ be a tribe.

A **homotopy** $h : f \rightsquigarrow g$ between two maps $f, g : A \to B$ in $\mathcal{C}$ is a map $h : A \to PB$ such that $sh = f$ and $th = g$.



Theorem

The homotopy relation $f \sim g$ is a congruence on the arrows of $\mathcal{C}$.

The homotopy category

Let $\mathcal{C}$ be a tribe.

The **homotopy category** $Ho(\mathcal{C})$ is the quotient category $\mathcal{C}/\sim$.

A map $f : X \to Y$ in $\mathcal{C}$ is called a **homotopy equivalence** if it is invertible in $Ho(\mathcal{C})$.

Every anodyne map is a homotopy equivalence.

An object X is **contractible** if the map $X \to \star$ is a homotopy equivalence.

Brown fibration category

Definition

(Ken Brown) A **Brown fibration category** is a category $\mathcal{E}$ equipped with a fibration structure together with a class of **acyclic maps** such that:

- Every isomorphism is acyclic;
- The class of acyclic maps has the 3-for-2 property;
- Every morphism can be factored as an acyclic map followed by a fibration;
- The class of acyclic fibrations is stable under base change.

Theorem

A tribe is a Brown fibration category if the acyclic maps are the homotopy equivalences.

Forthcoming Phd thesis of Page North in Cambridge.

Weak equivalences of tribes

Definition

We say that a homomorphism of tribes $F : \mathcal{E} \to \mathcal{E}'$ is a **weak-equivalence** if the induced functor $Ho(F) : Ho(\mathcal{E}) \to Ho(\mathcal{E}')$ is an equivalence of categories.

Theorem

A map $f : A \to B$ in a tribe $\mathcal{E}$ is a homotopy equivalence if and only if the functor $f^\star : \mathcal{E}(B) \to \mathcal{E}(A)$ is a weak-equivalence of tribes.

Meta-fibrations

Definition

We say that a homomorphism of tribes $F : \mathcal{E} \to \mathcal{E}'$ is a **meta-fibration** if the following conditions are satisfied:

- F and $Ho(F)$ are iso-fibrations;
- F is full on sections of trivial fibrations;
- F is full on diagonal filler of AF-squares;
- F is full on AF-factorisations.

For example, if $\mathcal{E}$ is a tribe, then the functor $\partial_1 : \mathcal{E}^{(1)} \to \mathcal{E}$ is a meta-fibration.

The base change of a meta-fibration along a homomorphism of tribes is a meta-fibration.

A fibration category of tribes?

- Objects $\rightsquigarrow$ tribes
- morphisms $\rightsquigarrow$ homomorphisms of tribes
- fibrations $\rightsquigarrow$ meta-fibrations
- Acyclic morphism $\rightsquigarrow$ weak equivalences of tribes

Theorem
The base change of an acyclic meta-fibration along a homomorphism of tribes is an acyclic meta-fibration.

I was not able to prove that every homomorphism can be factored as an acyclic homomorphism followed by a meta-fibration. Stronger axioms are needed.

Simplicial tribes

A **simplicial tribe** is a category enriched over simplicial sets having the structure of a tribes + compatibility axioms.

Theorem

The category of simplicial tribes has the structure of a Brown fibration category.

There is a relation with the work of Karol Szumilo.

THANK YOU FOR YOUR ATTENTION!