# Complex algebras of tree-semilattices

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# Outline

- Complex algebras and BAOs
- Boolean semilattices
- Linear Boolean semilattices
- Representable Boolean semilattices
- Complex algebras of tree-semilattices

# Introduction

For a (meet) semilattice $(S, \cdot)$, the ***complex algebra*** is

$\mathrm{Cm}(S) = (\mathcal{P}(S), \cup, \cap, -, \emptyset, S, \cdot)$

where for $x, y \in \mathcal{P}(S)$ we define $x \cdot y = \{r \cdot s \mid r \in x, s \in y\}$

So a complex algebra is a **complete and atomic Boolean algebra**

with additional operation(s)

Such algebras are examples of **Boolean algebras with operators** (BAOs)

# Introduction

Can define the complex algebra of **any relational structure**

Important in **algebraic logic**: connect **relational** and **algebraic semantics** of (poly)modal logics

Some **interesting varieties** are generated by complex algebras:

- **representable relation algebras** (generated by complex algebras of **Brandt groupoids**)
- **modal algebras** (generated by complex algebras of **directed graphs**)

# Introduction

Determining an **axiomatization** for the variety of BAOs generated by the complex algebras of a class of structures can be an **interesting problem**

Hodkinson, Mikulas, Venema [2001] gave a **general approach**:

for a **recursively enumerably** axiomatized class of algebras, the class of **subalgebras of the complex algebras** has a **recursive axiomatization**

Consider the variety generated by **complex algebras of semilattices**

Has been studied in a manuscript and preprint by C. Bergman [1995] [2015]

**Theorem**

*[J. 2004] The variety generated by complex algebras of semi***groups** *is not finitely axiomatizable*

# Introduction

Recall, for a (meet) semilattice $(S, \cdot)$, the ***complex algebra*** is

$\mathrm{Cm}(S) = (\mathcal{P}(S), \cup, \cap, -, \emptyset, S, \cdot)$

where for $x, y \in \mathcal{P}(S)$ we define $x \cdot y = \{r \cdot s \mid r \in x, s \in y\}$

Note **same notation** for the semilattice meet and its lifted version

Usually write $x \cdot y$ simply as $xy$ and $xx = x^2$

Algebras embedded in these BAOs are called ***representable Boolean semilattices***

The variety generated by all complex algebras of semilattices is denoted by **RBSL**

**Problem**: Is RBSL **finitely axiomatizable**?

# Boolean semilattices

The larger variety **BSL** of ***Boolean semilattices*** is the class of algebras $(A, \vee, \wedge, -, 0, 1, \cdot)$ such that

- $(A, \vee, \wedge, -, 0, 1)$ is a **Boolean algebra**
- $(A, \cdot)$ is a **commutative semigroup** and
- $\cdot$ is a ***square-increasing operator***: $x \leq x^2$,
  $x(y \vee z) = xy \vee xz$, and $x0 = 0$

The last two identities imply that for a finite BSL, the operation $\cdot$ is determined by its value on atoms

# Boolean semilattices

A Boolean semilattice is ***integral*** if $xy = 0$ implies $x = 0$ or $y = 0$

True in all complex algebras of semilattices, hence also in all subalgebras

**[Bergman 2015]** A BSL is integral **iff** it is finitely subdirectly irreducible

All integral BSLs with $\leq 2$ elements (1 is an atom) and $\leq 4$ elements ($1 = a \vee b$):

$A_1$ $\begin{array}{c|c} \cdot & 1 \\ \hline 1 & 1 \end{array}$ $B_1$ $\begin{array}{c|cc} \cdot & a & b \\ \hline a & a & a \\ b & a & b \end{array}$ $B_2$ $\begin{array}{c|cc} \cdot & a & b \\ \hline a & a & a \\ b & a & 1 \end{array}$ $B_3$ $\begin{array}{c|cc} \cdot & a & b \\ \hline a & a & 1 \\ b & 1 & b \end{array}$

$B_4$ $\begin{array}{c|cc} \cdot & a & b \\ \hline a & a & 1 \\ b & 1 & 1 \end{array}$ $B_5$ $\begin{array}{c|cc} \cdot & a & b \\ \hline a & 1 & 1 \\ b & 1 & 1 \end{array}$ $B_6$ $\begin{array}{c|cc} \cdot & a & b \\ \hline a & a & b \\ b & b & 1 \end{array}$

# Representations by semilattices

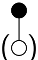
$A_1$

| $\cdot$ | 1 |
|---|---|
| 1 | 1 |

$= Cm(\circ)$

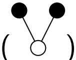
$B_1$

| $\cdot$ | $a$ | $b$ |
|---|---|---|
| $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ |

$= Cm($  $)$

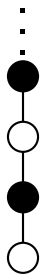
$B_2$

| $\cdot$ | $a$ | $b$ |
|---|---|---|
| $a$ | $a$ | $a$ |
| $b$ | $a$ | 1 |

$=$Cm(  )

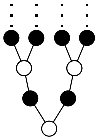
$B_3$

| $\cdot$ | $a$ | $b$ |
|---|---|---|
| $a$ | $a$ | 1 |
| $b$ | 1 | $b$ |

$= Cm($  $)$

$B_4$

| $\cdot$ | $a$ | $b$ |
|---|---|---|
| $a$ | $a$ | 1 |
| $b$ | 1 | 1 |

$= Cm($  $)$

$B_5$

| $\cdot$ | $a$ | $b$ |
|---|---|---|
| $a$ | 1 | 1 |
| $b$ | 1 | 1 |

$= Cm($homework$)$

$B_6$

| $\cdot$ | $a$ | $b$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $b$ | 1 |

$= Cm(?)$

Therefore $A_1, B_1, \ldots, B_5$ are in RBSL

Note that they are represented by **tree semilattices**

# Linear Boolean semilattices

A semilattice is ***linear*** if its partial order $\leq$ is a **chain**

(as usual $\leq$ is defined by $x \leq y \iff x = xy$)

The variety generated by **complex algebras of linear semilattices** is denoted by **LBSL**

## Theorem

[Bergman 1996, 2015] *LBSL is the variety of Boolean algebras with a commutative associative idempotent operator, i.e., square-increasing is strengthened to* $x = x^2$

## Proof.

(outline): Any subset $x$ of a linear semilattice satisfies $x = x^2$
Conversely, can assume $B$ is atomic (idempotence is canonical)
The set $A = \mathrm{At}(B)$ is linearly preordered by $\sqsubseteq$
Represent each block $C$ by a chain of cardinality $|C| + \omega$ $\square$

# Representable Boolean semilattices

## Lemma

*Algebras in RBSL also satisfy the following axioms:*

1. $x \wedge 1y \le xy$ (*we assume* $\cdot$ *has priority over* $\vee, \wedge$).
2. $x(xy - x) \le x^2 \vee (xy - x)^2$ [Bergman 1995]
3. $u \le yz \implies xu \le (xz \wedge v)y \vee (xz - v)u$
4. $xy \le x \vee y \implies x^2 \wedge y^2 \le xy$
5. $x \le yz \le x \vee y, x \vee w \le yw, x \wedge w = 0, xw \le x$ *and* $zw \le w \implies x \le y^3$

## Proof.

of 1. $x \wedge 1y \le xy$ holds in RBSL: let $p \in x$ and $p \in 1y$. Then $p = qr \le r$ for some $r \in y$. Hence $p = pr \in xy$. □

Note that 1. fails in $\begin{array}{c|cc} \cdot & a & b \\ \hline a & a & b \\ b & b & 1 \end{array}$ with $x = a$, $y = b$. Hence $B_6 \notin$ RBSL

# Representable Boolean semilattices

Let $B$ be a Boolean semilattice that satisfies the identity $x \wedge 1y \leq xy$

Define the relation $\sqsubseteq$ by $\quad x \sqsubseteq y$ if and only if $x \leq xy$

This relation is **reflexive** since $x \leq x^2$ and

**transitive**: $x \leq xy$ and $y \leq yz$ implies $x \leq xyz \leq 1z$, hence $x \leq xz$

Therefore $\sqsubseteq$ is a **preorder**

If $B$ is **atomic** (in particular, if it is finite) we restrict $\sqsubseteq$ to the **atoms** $\mathrm{At}(B)$

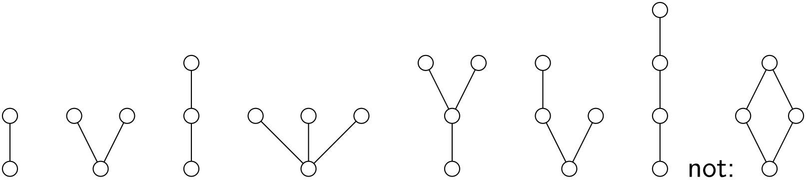
**Lemma**

*Suppose $B$ is embedded in $Cm(S)$ for a semilattice $S$*
*Then $S$ is finite* ***iff*** *$\sqsubseteq$ is a partial order on $At(B)$*

# Tree-representable Boolean semilattices

A semilattice is a ***tree-semilattice*** if its partial order is a **tree**

i.e., has a **bottom** and every **principal downset** is a **chain**




A BSL is ***tree-representable*** if it is embedded in the complex algebra of a tree-semilattice

The variety generated by **tree-representable Boolean semilattices** is denoted by **TBSL**

It properly contains **LBSL** and is a proper subvariety of **RBSL**

# Tree-representable Boolean semilattices

## Theorem

*Let $B$ be a finite Boolean semilattice that satisfies $x \wedge 1y \leq xy$ and assume $\sqsubseteq$ is a partial order on the atoms of $B$. Then the following are equivalent.*

1. *$B$ is* ***tree-representable***
2. *$B$ is embedded in the complex algebra of a* ***finite*** *tree-semilattice*
3. *$B$ satisfies the identity $(x \wedge xy)z \wedge -x \leq yz$*

## Proof.

(outline): 1.⇒3. Assume $B$ is a subalgebra of $\mathrm{Cm}(T)$ for some tree-semilattice $T$. Let $p \in$ LHS. Then $p \notin x$ and $p = qr$ s.t. $q \in x$, $r \in z$ and $q = uv$ where $u \in x, v \in y$. If $q \leq vr$ then $q = qvr = pv = p$, contradicting $p \notin x$. Hence $vr \leq q$, so $vr = vrq = qr = p \in yz$. 3.⇒2. Use the partial order $\sqsubseteq$ to build the tree-semilattice from the bottom up. The identity from 3. is used to show that $\cdot$ in $B$ is compatible with the meet operation on the tree-semilattice. 2.⇒1. holds by definition. □

# Further remarks on Boolean semilattices

It is **conjectured** that the variety TBSL is axiomatized relative to BSL by the identity $(x \wedge xy)z \wedge -x \leq yz$

A computer calculation shows that there are (up to isomorphism) 79 **integral** Boolean semilattices with 8 elements that satisfy **all 5 quasiequations** in the previous Lemma

Of these, 13 have $\sqsubseteq$ as a partial order on the atoms

They are all embedded in the complex algebra of some finite semilattice, hence in RBSL

There are 25 that satisfy the identity $(x \wedge xy)z \wedge -x \leq yz$ (including 9 from the previous 13)

It is not difficult to check that they are also in RBSL

Many of the remaining 50 algebras are known to be in RBSL as well, but currently this is not known for all of them

# Representation by disjoint union of semilattices

| $\cdot$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | 1 | 1 |
| $b$ | 1 | $b$ | 1 |
| $c$ | 1 | 1 | $c$ |

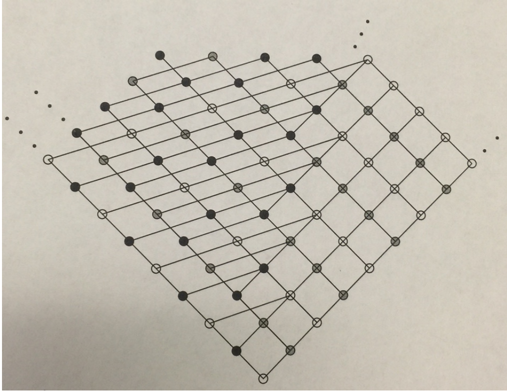
$= Cm(?)$

$(x \wedge xy)z \wedge -x \leq yz$ fails

Find 3 disjoint semilattices

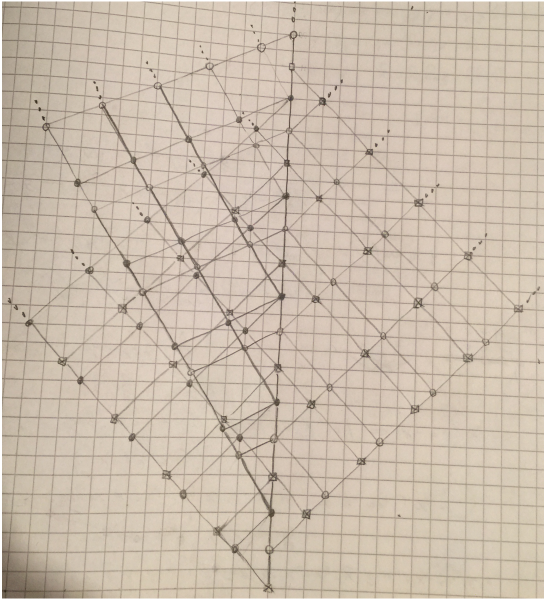
$A, B, C \quad$ s. t. $\quad A \cup B \cup C = S$

and $A \cdot B = A \cdot C = B \cdot C$



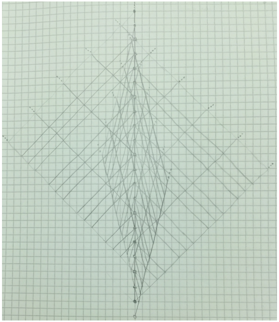
Miklos Maroti [2000]

# Another rendering of Miklos' example



# Extending to $n$ atoms

| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | 1 | 1 | 1 |
| $b$ | 1 | $b$ | 1 | 1 |
| $c$ | 1 | 1 | $c$ | 1 |
| $d$ | 1 | 1 | 1 | $d$ |



## Theorem

*The Boolean semilattice with atoms $a_1, a_2, \ldots, a_n$ and products $a_i \cdot a_i = a_i$ and $a_i \cdot a_j = 1$ if $i \neq j$ is representable by a semilattice that is the union of $n$ "spiraling half-planes".*

# References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Thank you