Classification of absorbent-continuous, complete and densely ordered, group-like FL_e-chains

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September 2014, Budapest

"in memoriam Franco Montagna"

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S. Jenei
Classification of absorbent-continuous, densely ordered, complete, group-like FL_e-chains (submitted)

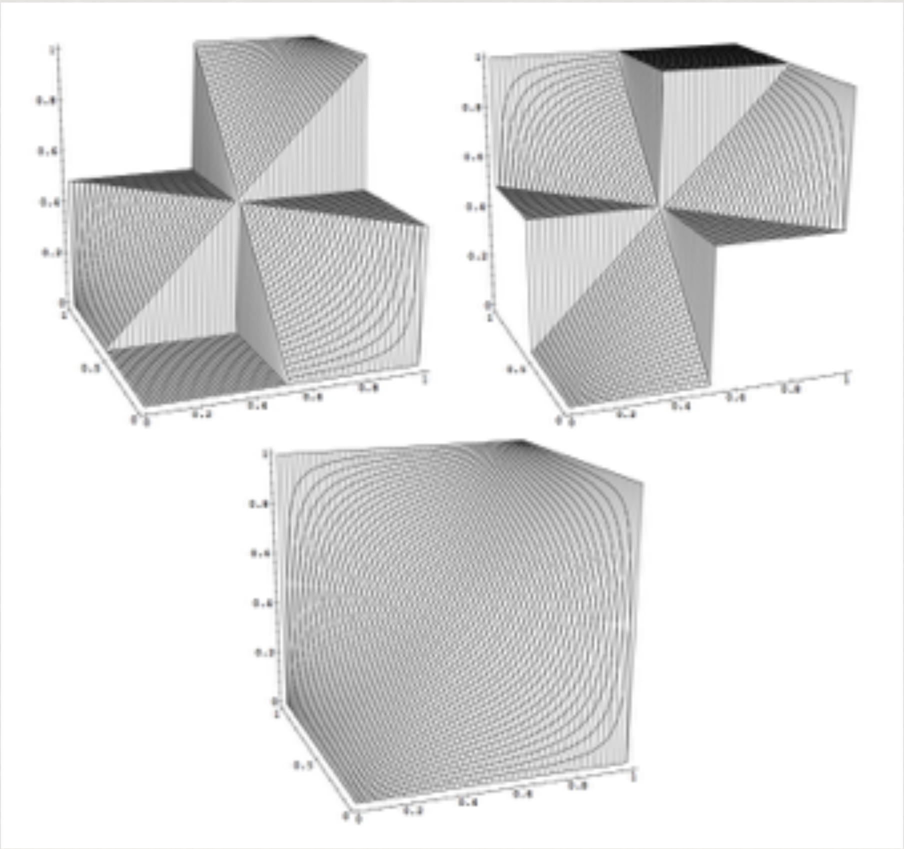
- ***Main Theorem***

  U is an absorbent-continuous, group-like FL_e-algebra on a ***complete****,* ***order dense*** *chain, with involution ' if and only if U is the twin-rotation of a BL-algebra and its de Morgan dual with respect to ', where the BL-algebra has components, which are either cancellative or MV-algebras over two elements, and the BL-algebra cannot have two consecutive cancellative components.*

Ordinal sums

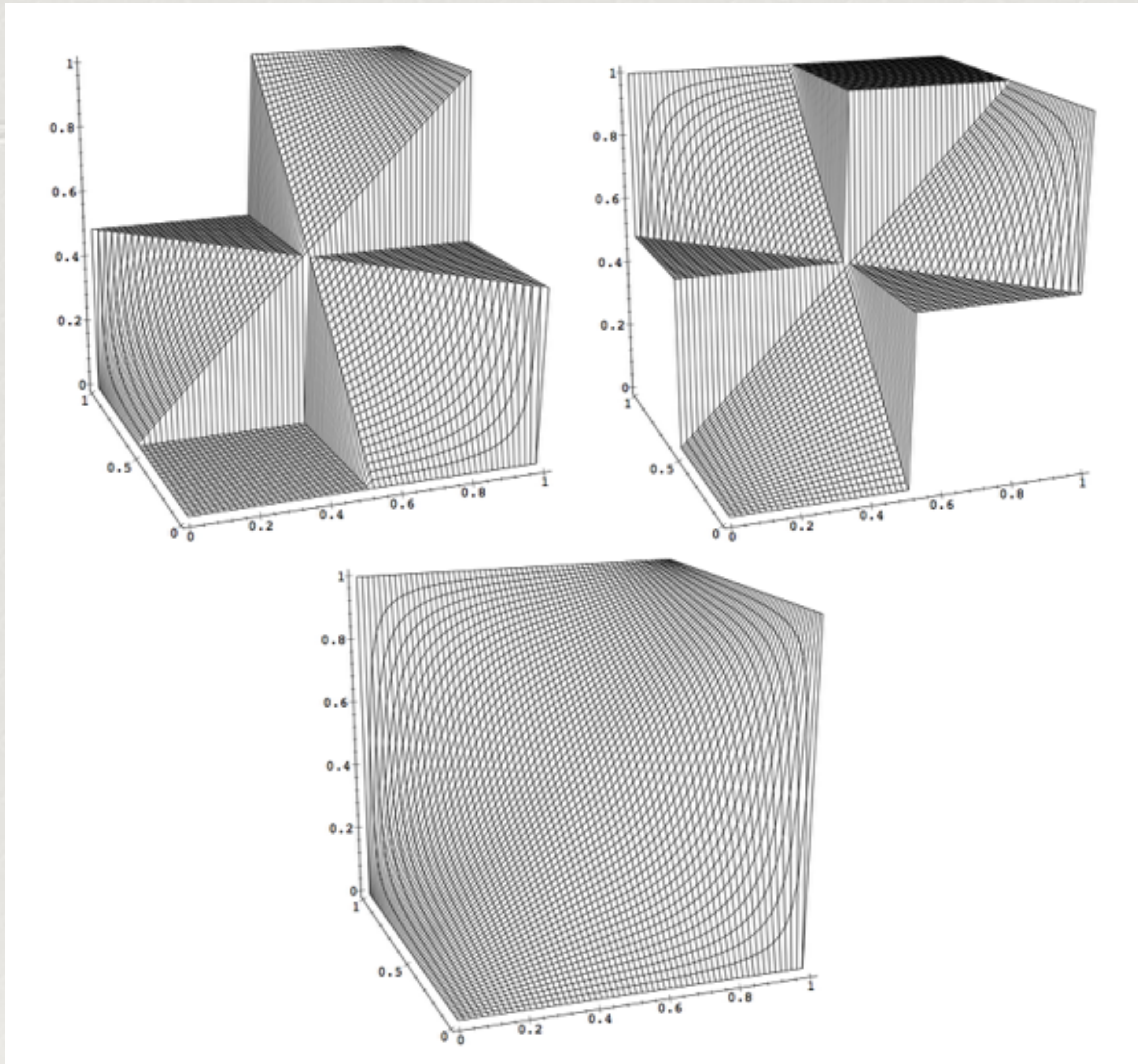
- *P. Aglianó, F. Montagna, Varieties of BL-algebras I: general properties, Journal of Pure and Applied Algebra, 181 (2–3), 2003, 105–129*
- *Absorbent continuity : for* $x \in X^-$, $a(x) \otimes x = x$, *where* $a(x) = \inf\{u \in X^- : u \otimes x = x\}$

Twin rotation

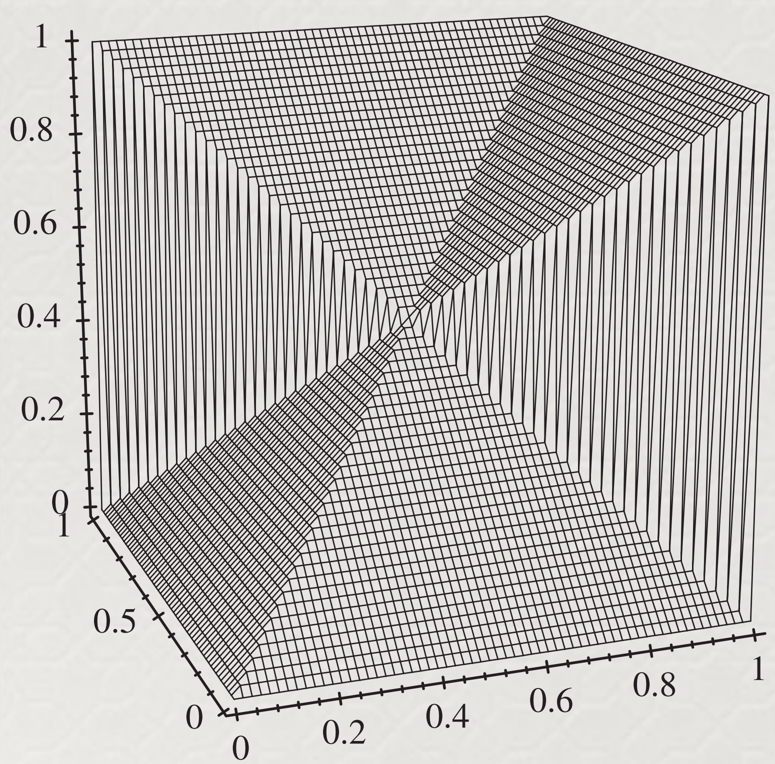


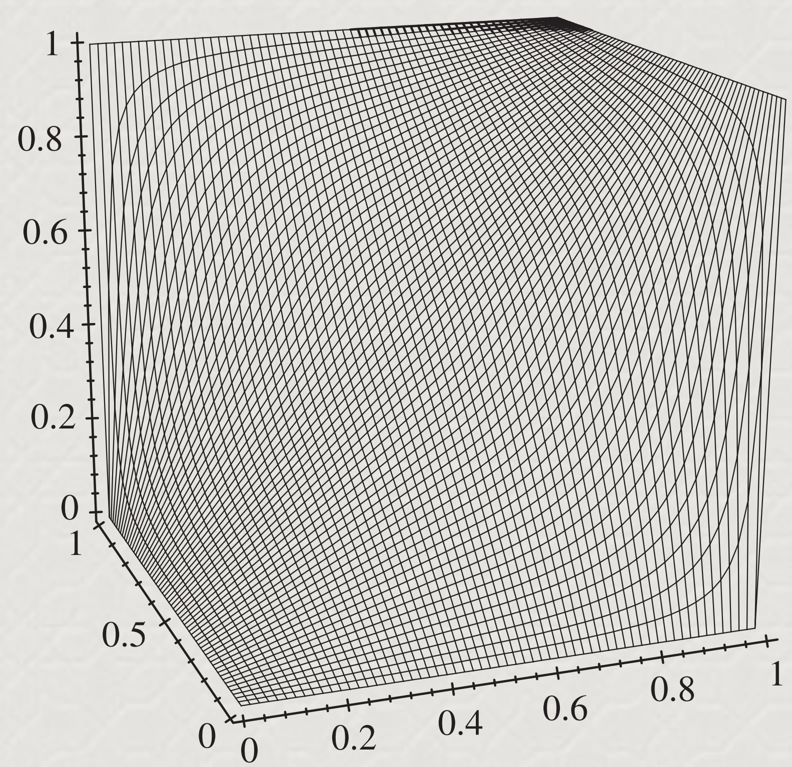
S. Jenei Sándor, H. Ono, On involutive FL_e-monoids
Archive for Mathematical Logic, 51:(7-8) pp. 719-738. (2012)

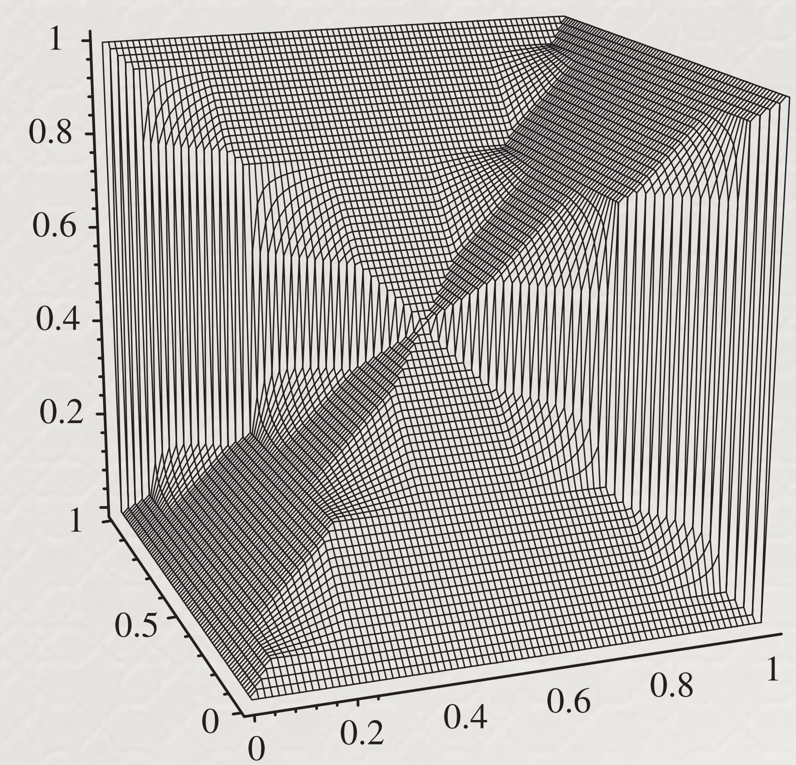
Twin rotation



Absorbent-continuous, complete, order-dense, group-like FL_e-chains







Classifications of residuated lattices

- *O. Hölder, Die Axiome der Quantität und die Lehre vom Mass, Berichte über die Verhandlungen der Königlich Sachsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physische Classe, 53 (1901), 1–64*
- *J.Aczél, Lectures on Functional Equations and Their Applications, Academic Press, New York-London, 1966.*
- *A. H. Clifford: Naturally totally ordered commutative semigroups, Amer. J. Math., 76 vol. 3 (1954), 631–646.*
- *P.S. Mostert, A.L. Shields, On the structure of semigroups on a compact manifold with boundary, Ann. Math., 65 (1957), 117–143*

Classifications of residuated lattices

- *P. Jipsen, F. Montagna, Embedding theorems for classes of GBL-algebras, Journal of Pure and Applied Algebra, 214 vol. 9 (2010), 1559–1575*
- *S. Jenei, F. Montagna, Strongly Involutive Uninorm Algebras, Journal of Logic and Computation 23:(3) pp. 707-726. (2013)*
- *S. Jenei, F. Montagna: A classification of certain group-like FL_e-chains, Synthese,(2014). doi:10.1007/s11229-014-0409-2 (selected papers from Logic and Relativity 2012 honoring István Németi's 70th birthday)*



S. Jenei, F. Montagna, A classification of certain group-like FLe-chains, SYNTHESE (2014) doi:10.1007/s11229-014-0409-2 selected papers from LR12 honoring István Németi's 70th birthday

- ***Theorem***

  U is an absorbent-continuous, group-like FL_e*-algebra on a* ***subreal*** *chain with involution ' if and only if U is the twin-rotation of a BL-algebra and its de Morgan dual with respect to ', where the BL-algebra has components, which are either cancellative or MV-algebras over two elements, and the BL-algebra cannot have two consecutive cancellative components.*

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- *U is an absorbent-continuous, group-like FL_e-algebra on a* ***subreal*** *chain with involution '*

Definition 2 An order dense chain is said to be *subreal* if its MacNeille completion is weakly real.

Definition 1 We call a chain $\langle X, \leq \rangle$ *weakly real* if

1. X is order dense,
2. there exists a dense $Y \subset X$ with $|Y| < |X|$, and
3. if x and y are different from the greatest element of X then for any $x, y \in Y$ there exist $u, v \in Y$ such that $u > x$, $v > y$, and there exists a strictly increasing function from $[x, u]$ into $[y, v]$.

S. Jenei
Erratum to "A classification of certain group-like FLe-chains"

- ***Theorem***

  U is an absorbent-continuous, group-like FL_e-algebra on a ***complete, weakly real*** *chain ...*

A false lemma

Lemma 9 *The MacNeille completion $\hat{\mathscr{U}} = \langle \hat{X}, \hat{\circledast}, \leq, t, f \rangle$ of an absorbent-continuous, subreal, group-like FL_e-chain $\mathscr{U} = \langle X, \circledast, \leq, t, f \rangle$ is an absorbent-continuous, weakly real, group-like FL_e-chain.*

Proof Denote by $\hat{A}$ the absorbent function of $\hat{\mathscr{U}}$. It suffices to prove that, for every $a \leq t$, $a \hat{\otimes} (\hat{A}(a))' = x$. Now, by Lemma 7, $\hat{A}(x)$ is weakly decreasing, and hence $\hat{A}(x)'$ is weakly increasing in the negative cone of $\hat{\mathscr{U}}$. Moreover, $a = \sup(Z)$ for some $Z \subset X$. By absorbent-continuity of $\mathscr{U}$, it follows that $a \otimes \hat{A}(a)' = \sup\{z \otimes A(z)' : z \in X\} = \sup\{z : z \in X\} = a$, and the claim is settled. $\square$

- *The proof is based on a claim that Dedekind-MacNeille completion of an absorbent-continuous, densely-ordered, group-like FL_e-chain is again absorbent-continuous. A counterexample can be constructed as follows. Let Z × R be the lexicographic product of the totally-ordered additive groups of integers and real numbers (hence $\langle p, q\rangle \leq \langle r, s\rangle$ iff $p < r$ or $p = r$ and $q \leq s$). This group forms an absorbent-continuous, densely-ordered, group-like FLe-chain which is not complete. Consider its DM-completion C. Then the elements of C are downsets (i.e., downward closed subsets of Z × R) which are either principal or have no supremum in Z × R. One such non-principal downset (in fact the greatest one in the negative cone) is $x=\{\langle p,q\rangle \in Z\times R \mid p\leq -1,\ q\in R\}$. If C would be absorbent continuous, then the set $S_x = \{z \in C \mid xz = x\}$ would have a minimum. However, this is not the case because $S_x = \{\downarrow\langle 0, r\rangle \mid r \in R\}$, where $\downarrow\langle 0, r\rangle$ is the principal downset generated by $\langle 0, r\rangle$.*

S. Jenei
Erratum to "A classification of certain group-like FL_e-chains"

- ***Theorem***

  U is an absorbent-continuous, group-like FL_e-algebra on a ***complete, weakly real*** *chain with involution ′ if and only if U is the twin-rotation of a BL-algebra and its de Morgan dual with respect to ′, where the BL-algebra has components, which are either cancellative or MV-algebras over two elements, and the BL-algebra cannot have two consecutive cancellative components.*

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Erratum to "A classification of certain group-like FL_e-chains"

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U is an absorbent-continuous, group-like FL_e-algebra on a ***complete, weakly real*** *chain with involution ' iff ...*

Definition 1 We call a chain $\langle X, \leq \rangle$ *weakly real* if

1. X is order dense,
2. there exists a dense $Y \subset X$ with $|Y| < |X|$, and
3. if x and y are different from the greatest element of X then for any $x, y \in Y$ there exist $u, v \in Y$ such that $u > x$, $v > y$, and there exists a strictly increasing function from $[x, u]$ into $[y, v]$.

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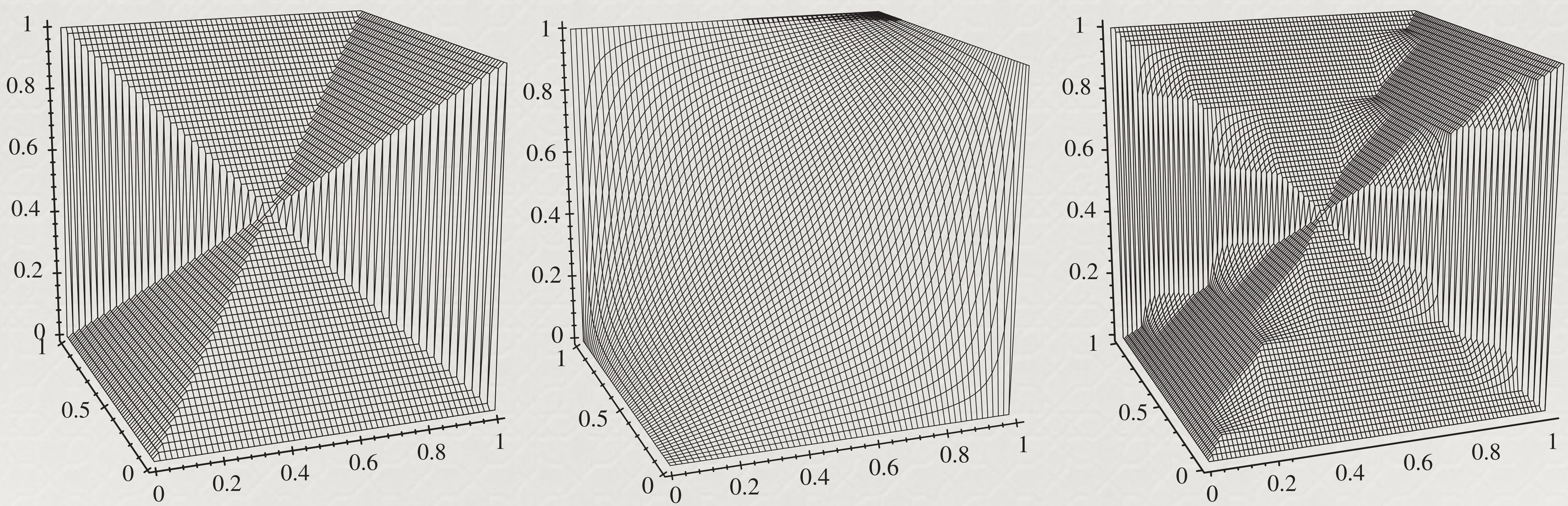
Classification of absorbent-continuous, densely ordered, complete, group-like FL_e-chains (submitted)

- ***Main Theorem***

  U is an absorbent-continuous, group-like FL_e-algebra on a complete, order dense chain, with involution ' if and only if U is the twin-rotation of a BL-algebra and its de Morgan dual with respect to ', where the BL-algebra has components, which are either cancellative or MV-algebras over two elements, and the BL-algebra cannot have two consecutive cancellative components.

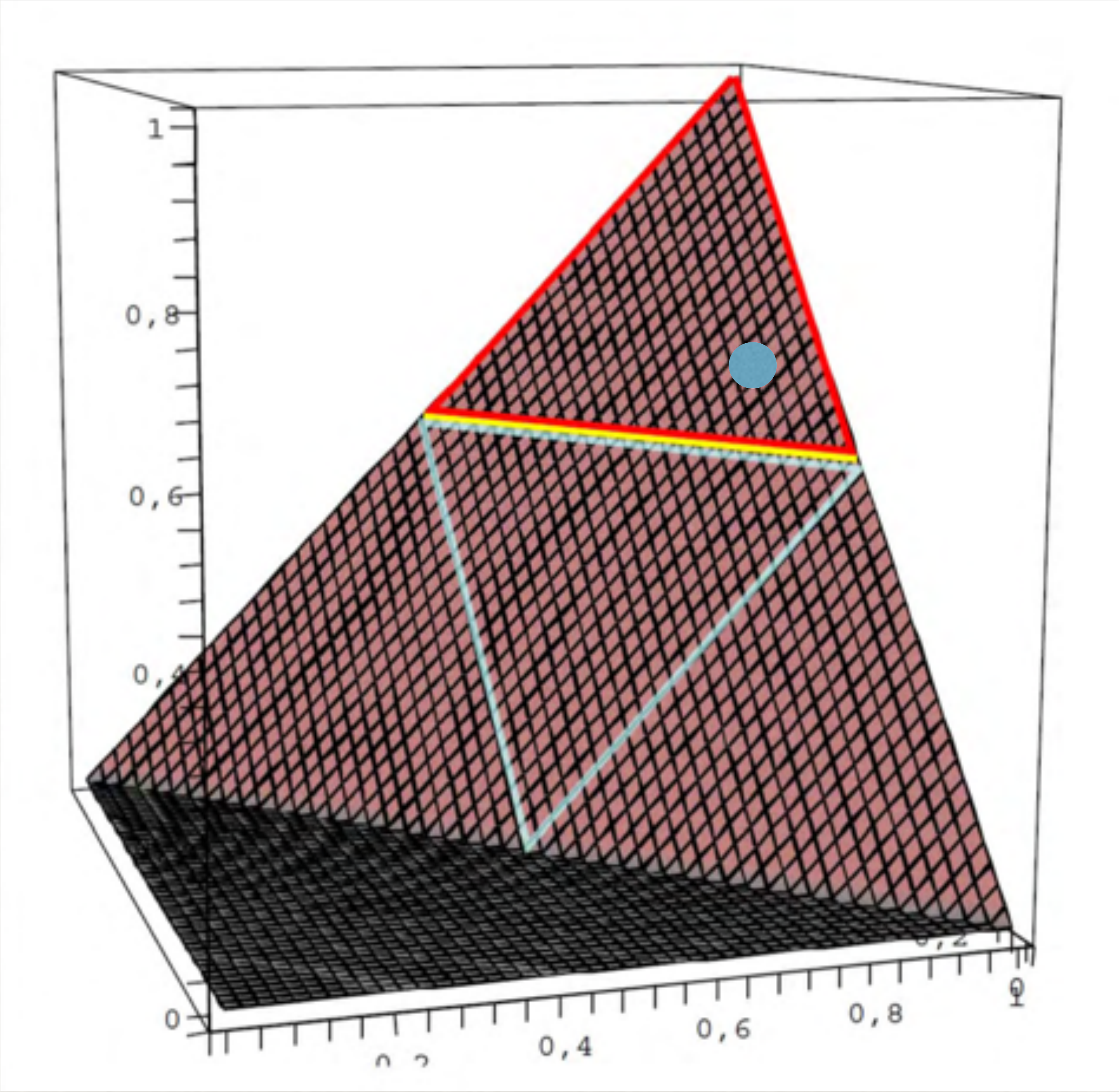
Absorbent-continuous group-like FL_e-chains over complete, order-dense chains



Geometric aspects of associativity

- *S. Jenei, On the geometry of associativity, Semigroup Forum 74:(3) pp. 439-466. (2007)*
- *S. Jenei, On the reflection invariance of residuated chains, Annals of Pure and Applied Logic 161:(2) pp. 220-227. (2009) S. Jenei, Erratum to "On the reflection invariance of residuated chains" [Ann. Pure Appl. Logic 161 (2009) 220-227], Annals of Pure and Applied Logic 161:(12) pp. 1603-1604. (2010)*

Main tool 1: Reflection-invariance



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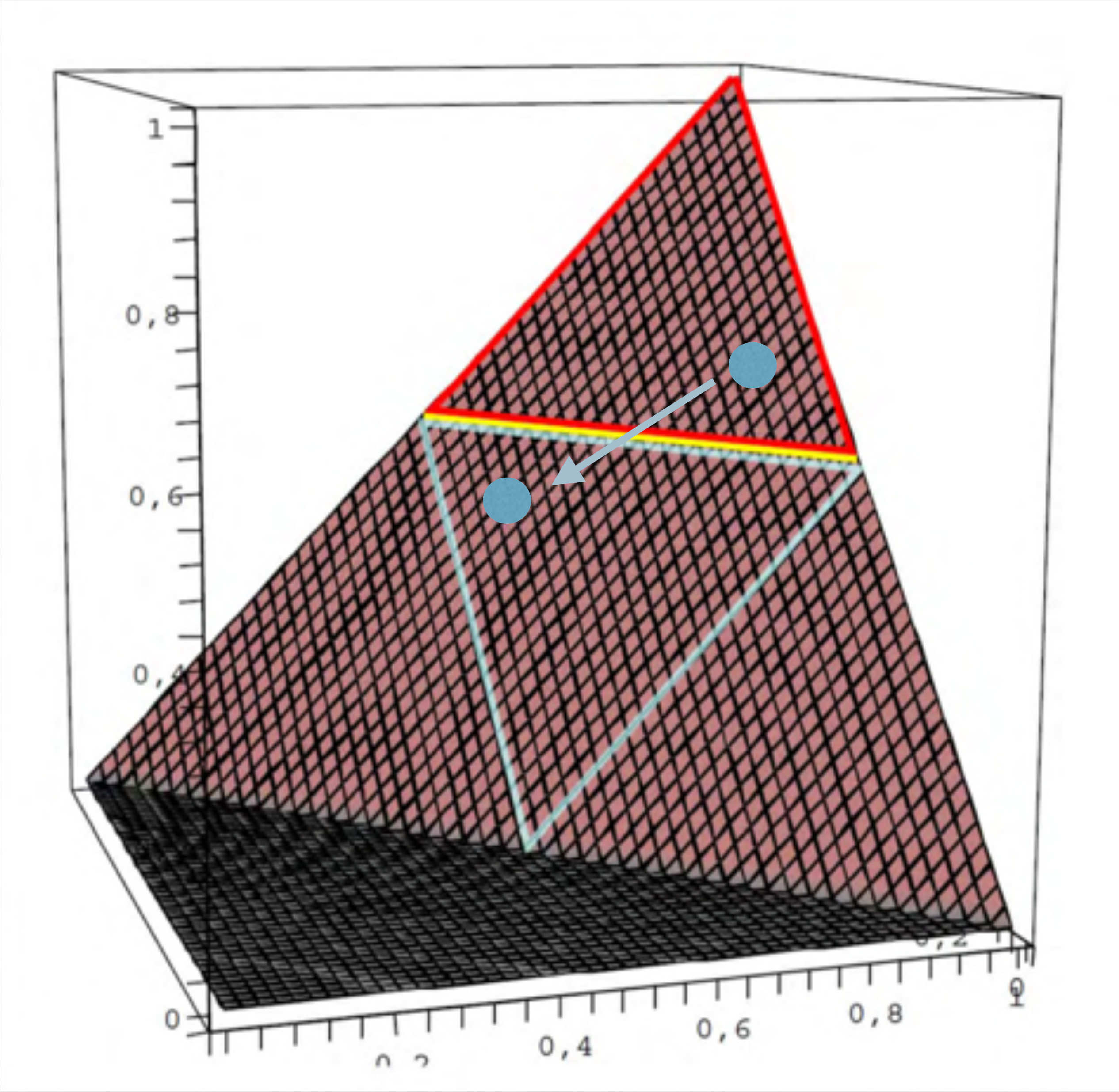
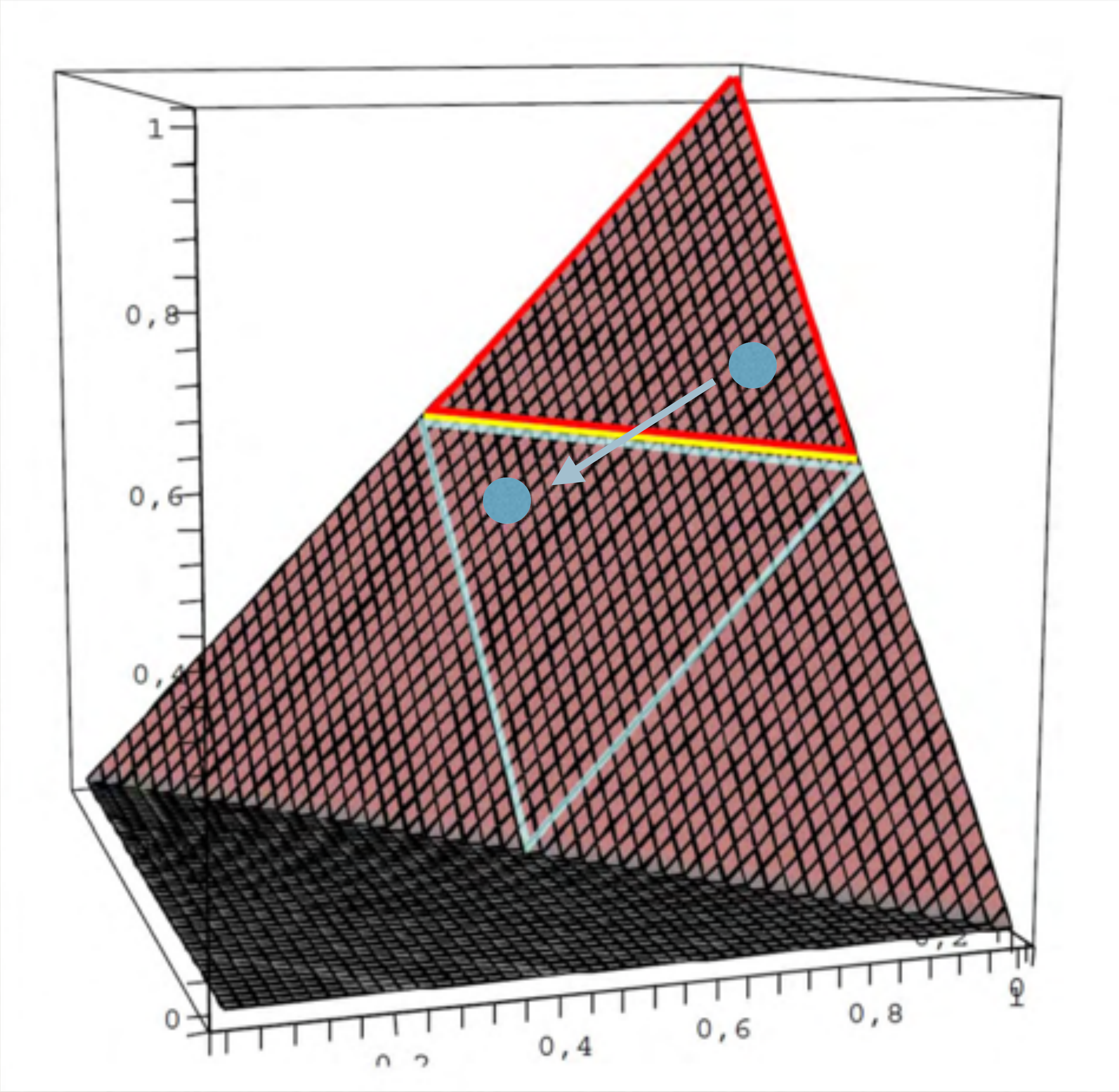


Main tool 1: Reflection-invariance

$x'_* y' = (x_* y)'$



Main tool 1: Reflection-invariance

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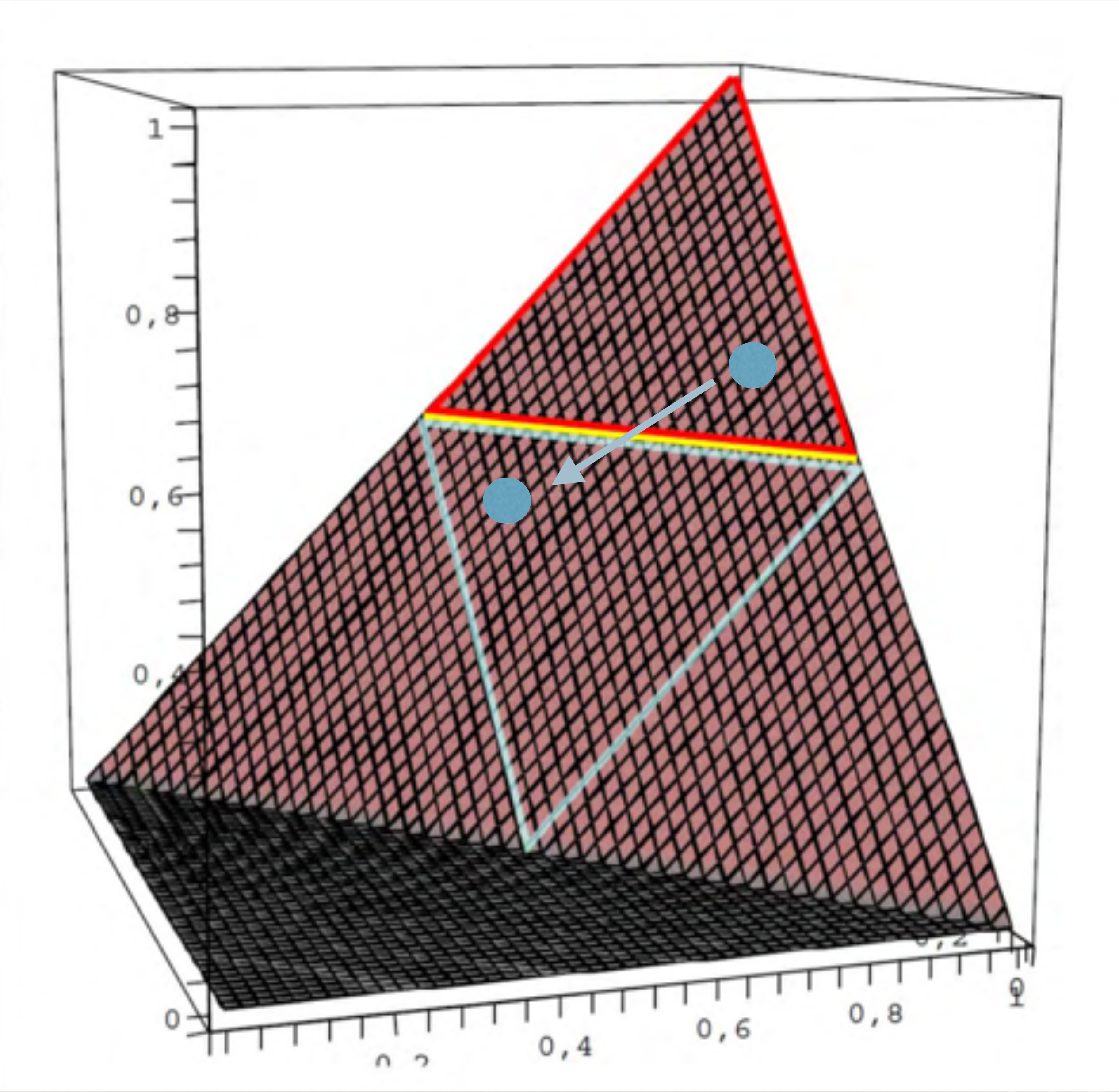

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Classification of absorbent-continuous, densely ordered, complete, group-like FL_e-chains (submitted)

- ***Main Tool*** $(x' * y')' = x * y$

Lemma 3 (Reflection Lemma) *Let $(X, \wedge, \vee, \circledast, \rightarrow_{\circledast}, t, f)$ be a group-like FL_e-algebra over a complete, order-dense chain. For $\top \neq x, y \in X$,*

$$(x' \circledast y')' = x \circledast_{co} y = x \circledast_Q y.$$

Definition 2 For a partially-ordered groupoid $(X, \leq, \circledast)$ over a complete lattice and for $x, y \in X \setminus \{\top\}$ define

$$x \circledast_{co} y = \inf\{x_1 \circledast y_1 \mid x_1 > x, y_1 > y\},$$
$$x \circledast_Q y = \inf\{x \circledast y_1 \mid y_1 > y\}.$$

Thank you for your attention!