Comparing presentations of algebraic theories

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Categories of equational theories

Equational theory
$T = (L, A)$

- $L = \bigcup_n L_n$ - set of operations,
- $t(\vec{x}^n) : X$ - term (over L) in a context X, $\vec{x}^n = x_1, \ldots, x_n \subseteq X$,
- $s(\vec{x}^n) = t(\vec{y}^n) : X$ equations in contexts,
- A is a set of equations in finite contexts $\vec{x}^n$, $n \in \omega$.

Interpretations: morphisms of equational theory
$I : T \to T' = (L', A')$

- $I_n : L_n \to \mathcal{T}r(L', \vec{x}^n)$, for $n \in \omega$ functions,
- extends to $\bar{I}_n : \mathcal{T}r(L, \vec{x}^n) \to \mathcal{T}r(L', \vec{x}^n)$,
- $A' \vdash \bar{I}(t) = \bar{I}(s) : \vec{x}^n$ for $t = s : \vec{x}^n$ in A.

We identify two such interpretations if they are provably equal.

Restrictions:

- shape of equations: $s = t : \vec{x}^n$
  - regular: all variables from $\vec{x}^n$ occur both sides,
  - linear-regular: all variables from $\vec{x}^n$ occur on both sides exactly once,
  - rigid: linear-regular and if $A \vdash t(x_1, \ldots, x_n) = t(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \vec{x}^n$, then $\sigma = id_{\underline{n}}$.
- shape of interpretations: $L_n \ni f \mapsto t : \vec{x}^n$
  - regular: all variables form $\vec{x}^n$ occur in t,
  - linear-regular: all variables form $\vec{x}^n$ occur in t exactly once,
  - rigid: as above.

We have categories: **RiET** $\to$ **LrET** $\to$ **RegET** $\to$ **ET**.

Algebraic presentations of categories of equational theories

Equivalent descriptions of the category **ET** and its subcategories...

For objetcs (theories) we need to specify:

- Data: some terms in some contexts modulo equations (e-terms)
- Operations:
  - composition (substitution of e-terms)
  - identities/projections/actions on e-terms

Morphisms (of theories) will be always obvious: just preserving the data involved.

Monads (on *Set*)
(T, η, μ):

- all terms in all contexts: $T(X)$ -set of e-terms in context X,
- monad composition: flattening terms on terms to terms:

$$\mu_X : T^2(X) \to T(X),$$

- projections:

$$\eta_X : X \to T(X)$$
$$x \mapsto x : X$$

- all actions: $f : X \to Y$ a function;

$$T(f) : T(X) \to T(Y)$$

  action of f on e-terms in context X,
- $+$ monad axioms.

Morphisms of monads are natural transformations $\tau : T \to T'$ compatible with η's and μ's.

Restrictions: categorical properties of T and natural transformations η and μ.

- Properties of T (finitary)

regular	T preserves pullbacks along monos (semi-analytic),
linear-regular	T preserves weak wide pullbacks (analytic),
rigid	T preserves wide pullbacks (polynomial).

- Properties of η and μ

regular	naturality square for monos are pullbacks (semi-cartesian),
linear-regular	naturality square for all functions are weak pullbacks (weakly-cartesian),
rigid	naturality square for all functions are pullbacks (cartesian).

We have categories of monads on *Set*:

$$\mathbf{PolyMnd} \to \mathbf{AnMnd} \to \mathbf{SanMnd} \to \mathbf{Mnd}.$$

Lawvere theories $\pi : \mathbb{F}^{op} \to \mathbf{T}$

- finite tuples of e-terms in contexts $\vec{x}^n$, $n \in \omega$,
- usual categorical composition,
- projections (images of π),
- $\mathbf{T}$ is a category, $\mathbb{F}$ is the category of finite sets $\{1, \ldots, n\}$ $n \in \omega$ and functions, π is bijective on objects and preserves products.

Morphisms of Lawvere theories are functors preserving projections.
LT - the category of Lawvere theories.

Restrictions: factorization systems on L with the right class orthogonal to a subclass of morphisms from $\mathbb{F}$, possibly with some additional properties.

- Factorization systems

  regular: the class of right orthogonal maps to the images of surjections under $\pi : \mathbb{F} \to \mathbf{T}$,

  linear-regular: the class of right orthogonal maps to the images of all functions under $\pi : \mathbb{F} \to \mathbf{T}$,

  rigid: as above and the functions

$$\rho_n : S_n \times Aut_{\mathbf{T}}(1)^n \longrightarrow Aut_{\mathbf{T}}(n)$$

$$(\sigma, a_1, \ldots, a_n) \mapsto a_1 \times \ldots \times a_n \circ \pi_\sigma$$

  are bijections.

The morphisms of the above Lawvere theories have to preserve the factorization systems (e.i. right and left classes).

We have categories of Lawvere theories:

$$\mathbf{RiLT} \to \mathbf{AnLT} \to \mathbf{RegLT} \to \mathbf{LT}.$$

Operads

- O_n some 'essential' e-terms in contexts $\vec{x}^n$, $n \in \omega$,
- operadic composition ('disjoint domain composition'): $O_{n_1} \times \ldots, O_{n_k} \times O_k \longrightarrow O_{\sum_i^k n_i}$,
- identity $\iota \in O_1$,
- actions: $\mathbb{F}(\underline{n}, \underline{m}) \times O_n \to O_m$

$$\langle f, t(x_1, \ldots, x_m) \rangle \mapsto t(x_{f(1)}, \ldots, x_{f(n)}),$$

- plus compatibility of actions with compositions.

Morphisms of $\mathcal{F}$-operads send n-ary operations to n-ary operations preserving identity, compositions, and actions.

FOp - the category of operads with action of $\mathbb{F}$.

Restrictions: are on the actions on operations.

- Actions

regular	we consider actions of surjections only,
linear-regular	(=symmetric operad) we consider actions of bijections only,
rigid	as above and the actions must be free.

The notions a morphism of operads remains the same but it means something else here, the operations in operads are NOT all term operations but only those that are 'essential' for the theory. They constitute the right classes of morphisms (having 1 as the codomain) in factorization systems of the corresponding Lawvere theories.

We have categories of operads:

$$\mathbf{RiOp} \to \mathbf{SOp} \to \mathbf{RegOp} \to \mathbf{FOp}.$$

(Abstract) Clones

- e-terms in contexts $\vec{x}^n$, $n \in \omega$,
- clonic composition ('same domain composition'): $O_n^k \times O_k \longrightarrow O_n$,
- projections π_i^n, $1 \leq i \leq n$.

Morphisms of clones send n-ary operations to n-ary operations preserving projections and compositions.

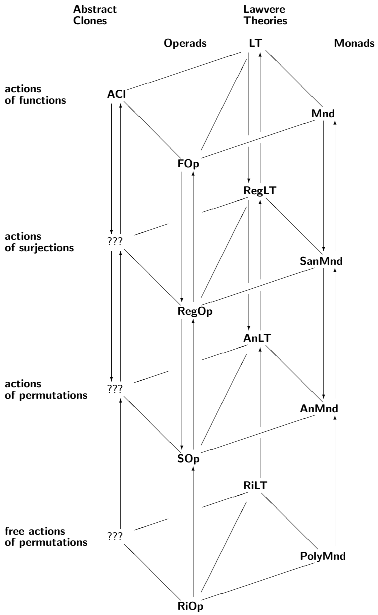
ACl - the category of clones.

Restrictions: ???.
One can import the conditions from the other presentations but the conditions look foreign to abstract clones...

Do we have categories of clones:

$$??? \to ??? \to ??? \to \mathbf{FOp}?$$

Algebraic Presentations of Equational Theories



Equational Theories

ET- all equational theories

RegET- regular equational theories

LrET- linear-regular equational theories

LrET- linear-regular equational theories

Thank You for Your Attention!